June, 1973

Report ESL-R-503
MIT DSR Projects 73935,
76265, & 72917

ON THE DECENTRALIZED CONTROL OF LARGE-SCALE SYSTEMS

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This report is based on the unaltered thesis of Chee-Yee Chong submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the Massachusetts Institute of Technology in May, 1973. This research was conducted at the Decision and Control Sciences Group of the M.I.T. Electronic Systems Laboratory, with partial support provided by the Air Force Office of Scientific Research under Grant AFOSR-72-2273, by the National Science Foundation under Grant GK-25781, and by the NASA Ames Research Center under Grant NGL-22-009-124.

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S.B., Massachusetts Institute of Technology
1969

S.M., Massachusetts Institute of Technology
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SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
May, 1973

Signature of Author. . . . . . Chee-Yee Chong . . . . . .
Department of Electrical Engineering, May 29, 1973

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Chairman, Departmental Committee on Graduate Students

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Submitted to the Department of Electrical Engineering on May 29, 1973 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The decentralized control of stochastic large-scale systems is considered. Particular emphasis is given to control strategies which utilize decentralized information and can be computed in a decentralized manner.

The deterministic constrained optimization problem is generalized to the stochastic case when each decision variable depends on different information and the constraint is only required to be satisfied on the average. For problems with a particular structure, a hierarchical decomposition is obtained.

For the stochastic control of dynamic systems with different information sets, a new kind of optimality is proposed which exploits the coupled nature of the dynamic system. The subsystems are assumed to be uncoupled and then certain constraints are required to be satisfied, either in a "off-line" or "on-line" fashion. For off-line coordination, a hierarchical approach of solving the problem is obtained. The lower level problems are all uncoupled. For on-line coordination, distinction is made between open loop feedback optimal coordination and closed loop optimal coordination. A hierarchical decomposition of the problem is possible in each case. The linear-quadratic-Gaussian problem is solved in detail for both off-line and on-line coordination. The resulting control strategies are found to have certain nice properties.

THESIS SUPERVISOR: Michael Athans

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ACKNOWLEDGEMENT

I would like to express my deep gratitude and appreciation for the guidance and encouragement given me by Professor Michael Athans, both related to my thesis and in more general matters pertaining to my professional career. I would also like to thank my thesis readers, Professors Sanjoy Mitter and Timothy Johnson for their valuable comments and suggestions.

Many members of the Electronics Systems Laboratory have contributed a lot to my education. In particular, I would like to thank Professor Leonard Gould, who has been my faculty councilor almost ever since I entered graduate school. Among my fellow students, I would like to mention in particular Keith Glover and Nils Sandell. Credit must also be given to Ann Hicks and the entire staff of the Publications Department for typing the final draft of this thesis, and to Arthur Giordani, Norman Darling and John Sherman for the drawings.

I would also like to thank Dr. Edison Tse, of Systems Control, Inc. for introducing me to the area of control and for his interest in my education ever since.

I am indebted to Professors Paul Kleindorfer and Murat Sertel of the Sloan School of Management for presenting me systems ideas from a management point of view.

This research was conducted at the Decision and Control Sciences group of the M.I.T. Electronic Systems Laboratory, with partial support provided by the Air Force Office of Scientific Research under grant AF-AFOSR-72-2273, by the National Science Foundation under Grant GK-25781, and by the NASA Ames Research Center under grant NASA-NGL-22-009-124.

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CHAPTER 1

INTRODUCTION

1. Nature of Large-Scale Systems

It is hard to give a precise definition of a large-scale system, nor do we believe that there is one. A definition based on the number of components in the system is unsatisfactory since this would include systems which we would normally not consider large, e.g., a heated rod. Rather, the largeness of such systems seems to reflect the effort required to understand and control them. The following features, though not exhaustive, seem to be characteristic of most large-scale systems.

(a) Large number of equations, usually coupled, describing the system.

(b) Large number of decision variables to be manipulated. Usually these decision variables can be collected into groups to be chosen by different agents according to their spatial configuration or their function.

(c) The decision variables and state variables are so distributed that the information available to agents in charge of the different groups of decision variables are different. This feature is usually absent in traditional small-scale control systems but is inevitable in large-scale systems. This kind of information pattern is sometimes termed nonclassical [W2].

(d) Presence of uncertainty. When uncertainty is absent, it would be able to exchange the total information

available among all the decision agents, thus rendering the information pattern classical.

(e) More than one preference ordering for the entire system. Two cases are possible. The same set of preference orderings may be shared by all the decision agents or each decision agent may have a different one. The first case has been studied under the topic of nonscalar performance criterion, e.g., ref. [H3]. The second case generally arises in a game.

(f) Difficulty in modelling the system. This can be illustrated by the effort spent in understanding systems such as a power system or the economic system.

These last two aspects are very important but they will not be considered in this thesis. Rather, we shall assume that a model of the system is known and there is one single preference ordering for the entire system which is represented by a cost functional (performance index). All the control agents choose their controls to optimize (minimize) this cost functional. We feel that the problem of controlling a large-scale system is complicated enough even without the last two features.

Two constraints which may be neglected in the control of small scale systems become extremely important when the system is large.

(a) Communication. It may be expensive or even technically impossible to provide good communication links between all the control agents.

(b) Computation. The sheer size of some typical large-scale systems, e.g., economic systems, may make the control

problem bigger than that can be handled by the fastest computers available. For the control of dynamic systems, we need actually to distinguish between two kinds of computation, off-line and on-line. Off-line computation is what can be computed before the system starts running, e.g., the computation of the optimal strategies. On-line computation has to be done in real time while the system is actually running, e.g., transforming the data received in real time into decisions (controls) using the optimal strategies computed off-line. In general, on-line computation presents bigger problems than off-line computation since it has to be done in real time.

Without these constraints, there would be little difference between the control of large-scale and small-scale systems. The information available to the control agents can be pooled together and the optimal control policy solved for like a small-scale problem. This policy can then be dispatched to the control agents and implemented. The constraints on communication and computation make this approach of centralized control impossible. Some form of decentralization is therefore necessary. This is the central issue in the control of large-scale systems.

Another advantage of decentralization which is related to communication is reliability. A design based on centralization cannot function properly if the communication links between the central agency and the subsystems fail. On the other hand, decentralized control has the nice property that a certain degree of autonomy is retained for each subsystem.

Thus even though no signals are received from the central agency, some form of optimality is still possible.

2. Historical Survey

This design and control of large-scale systems has become a very popular research area in system and control theory. Previously, this problem was investigated mainly by economists and management scientists who have to deal with systems much larger than those encountered by engineers. The design of management information systems and decentralization by price mechanisms in organizations can all be regarded as methods of controlling large-scale systems [A5]. On the other hand, engineers do have some experience with large-scale systems, e.g., the power system which is more or less controlled in a hierarchical manner [S1].

Roughly speaking, past efforts on the control of large-scale systems can be summarized into four categories.

(1) Resource allocation processes. These deal with a special class of static systems called the economy. Given their initial resource endowments, their production possibilities and preferences, the economic units or participants of the economy are to choose their production and exchange activities such that a pareto-optimal point is reached.

Let

$I = \{1,\dots\dots\dots,n\}$: the set of economic units

X: the commodity space

0_X: the identity element of X

${}^iX^1 = {}^iX^2 = X$ for all $i \in I$

$Y^i = {}^iX^1 \times {}^iX^2$ for all $i \in I$

$Y = Y^1 \times \dots\dots\dots \times Y^n$

For any $d^i \varepsilon\ {}^iX^1$, $z^i \varepsilon\ {}^iX^2$, the pair $(d^i, z^i) = s^i \varepsilon\ Y^i$ is called an economic plan of the ith unit.

d^i: exchange activities

z^i: production activities

$s = (s^1,\ldots.s^i,\ldots.,s^n)$: program

The ith component of the economic environment is defined as the triple

$$({}^iA, w_o^i, {}^iR) = e^i$$

where

${}^iA$ is a non-empty subset of Y; the set of i-achievable programs.

w_o^i is an element of X; the initial resource endowment of the ith unit.

${}^iR$ is a total ordering of the elements of ${}^iA$, i.e., ${}^iR$ is a transitive, reflexive, connected relation defined on ${}^iA$.

The economic environment is then defined as

$$e = (e^1,\ldots\ldots,e^i,\ldots\ldots,e^n)$$

Given an economic environment, as adjustment process is a set of rules for exchanging information among the economic units, regarding their components of the environment, in order to reach an agreement about the economic program to be implemented. Formally, an adjustment process π is defined as

$$\pi = (L, f, \phi)$$

where

(a) L is the set of messages that the units can use for exchanging information.

(b) $f = (f^1, \ldots\ldots f^n)$ is the n "response" functions. $f^i : L^n \times E \rightarrow L$ where E is the class of environments.

(c) ϕ is the outcome rule which associates with each equilibrium message an outcome set $\bar{S}$ of economic programs.

Informational decentralization requires that the response function of each unit depends on the environment e only through its own components. Hurwicz [H4] and Camacho [C6] have presented different informationally decentralized adjustment processes whose equilibria are Pareto-optimal for different economic environments. It should be noted that this class of problems involve static, deterministic systems of a very special nature. However, the decision making is also decentralized since the economic units do not have to get together to choose their strategies.

(2) Team decision problems. This class of problems is first proposed by Marschak [M3]. There is a single objective function and a number of decision makers each with different information on the state of the system. Optimal decision rules transforming the information into action are required. The scheme is informationally decentralized but the decision makers have to find their decision rules together.

The linear-quadratic-Gaussian static case has been considered by Radner [R1]. It is found that the optimal decision rules are linear. Reference [M5] contains most of the original work done. For the dynamic case, when one decision maker's information depends on the action of another decision maker, the situation is more complicated. We are used to the case when the decision maker's information includes that of all the decision makers who act before him. Under those circumstances, the optimal decision rules are linear and are in fact given by the "Separation Theorem" [A2, M2, W4]. However, Witsenhausen showed by a counter-example that the optimal control strategies need not be linear, [W3], contrary to the solution of ordinary linear-quadratic-Gaussian problems. He also studied when the Separation Theorem holds for problems with non-classical information pattern [W2]. Ho and Chu [H3] gave conditions on the information structure such that the optimal decision rules are linear. Chong and Athans [C3] showed that the advantage of decentralized information in team decision problems may be offset by the additional complicated computation required to find the optimal strategies. Aoki [A1] studied a dynamic team when the decision agents involved are allowed to share information about their past control values.

(3) Hierarchical systems. The decision agents controlling the system are arranged in a hierarchy of levels. Each agent in a level communicates with several agents in the level under it

and with one agent in the level above. The agents at different levels perform different functions. The agents at the lowest level actually interact with the system under control while those above act as information processing centers or make long term decisions. Although there are a lot of intuitive advantages of having a hierarchical system [S2], such as reliability, adaptibility and ease of computation, very little is available in the form of a mathematical theory [M1]. Most of the work done in the hierarchical decomposition of systems has been inspired by mathematical programming. One version is the following. The subsystems in the large system are controlled as isolated units. This would be unsatisfactory since the subsystems are actually coupled to each other. The coupling effect is taken care of by a coordinator who sends out coordinating signals to the lower level controllers. The coordinating signals are so chosen such that the overall objective of the system is achieved. The original control problem is thus divided into two levels. The lower level problem consists of independent optimization problems dependent on the coordinating signal. The higher level problem is that of the coordinator. Much of the work done in the decomposition of mathematical programming can be found in [L3] and [W5]. It should also be noted that some of this work is actually related to resource allocation processes. In general, hierarchical decomposition methods motivated by mathematical

programming deal mainly with the computational aspects of the problem. The presence of uncertainty and the flow of information between subsystems is seldom treated.

(4) Controllability and stabilizability of decentralized dynamic systems. Although this work is not primarily concerned with optimization, it addresses itself to some very fundamental questions in large-scale systems, namely the controllability of decentralized systems and their feedback stabilizability. Preliminary work was done by McFadden [M4] who considered a system that arises in modelling certain aspects of economic systems where several national agencies exercise regulatory control power over different aspects of economic activities. Aoki [A4] considers the stabilizability of decentralized linear time-invariant dynamic systems with coordination and/or communication among control agents. It is found that controllability of the systems no longer implies stabilizability and the control agents must in general communicate with each other in order to stabilize the system by feedback.

3. Motivation

From the previous section we can see that most of the work done on large-scale systems treat the two important issues of computation and information separately. Work dealing with the computational aspect of the optimization of large-scale systems is almost exclusively deterministic. The flow of information within the system is therefore unimportant. On the other hand, work on decentralized information structure seldom considers the importance of computational requirements. Some of the optimal solutions to dynamic teams are computationally not feasible. Since in the actual control of large-scale systems, computational considerations are as important as those of information, decentralized information may not be as efficient as it may appear.

Decentralized information structure almost inevitably gives rise to more complicated decision rules than centralized information structure. This may be explained as follows. Since each decision agent has only partial a posteriori information about the state of the system, he may want to generate the missing information using the common a priori information available to him. Mathematically, he is required to extract whatever information that is available in order to be optimal.

This motivates us to use a broader interpretation about decentralized information. We shall consider two kinds of information: a priori information and a posteriori information. A priori information consists of structural information and performance indices. A posteriori information consists of measurements on the system.

Thus only a posteriori information may be decentralized, or both a priori and a posteriori information may be decentralized. If only a posteriori information is decentralized, then the common a priori information may induce each decision agent to use very complicated decision rules. If appropriate decentralization is chosen for both a priori and a posteriori information, the decision rules for the decision agents will be simple. There will be a severe loss of optimality, however, since the individual agents do not know that they are controlling the same system.

To compensate for the loss in optimality due to the decentralized information structure (both a priori and a posteriori) of the decision agents, we introduce a <u>higher level coordinator who possesses all the a priori information</u>. The coordinator may have a posteriori information about the system but in general this information is less detailed than that of the decision agents. The duty of the coordinator is to transmit coordinating parameters to the individual decision agents such that the system is coordinated in some sense.

We shall thus consider systems with a multilevel information structure. The higher level coordinator has all the a priori information and some a posteriori information. The lower level decision agents have decentralized a priori information as well as decentralized a posteriori information. They also receive certain coordinating parameters from the coordinator. A structure with many levels can be investigated, although in this thesis, only the two-level structure will be considered.

4. Structure of this Thesis

This thesis is structured in the following manner.

In Chapter 2 the decomposition for a static stochastic optimization problem is considered. The problem under consideration consists of several decision agents each having different and noisy information on the state of the system. There is also a coordinator who sees that certain constraints are satisfied with respect to his own information. Results in mathematical programming are used to obtain a hierarchical decomposition for this stochastic problem. It is shown that with the coordinating parameter transmitted from the coordinator, the lower level problems of the decision agents can be solved in a decentralized manner.

In Chapter 3, the concept of decentralized a priori information is used to obtain an off-line decomposition for nonlinear stochastic dynamic systems. The lower level controllers assume that they are controlling uncoupled dynamic systems with their decentralized a posteriori information. The coordinator has all the common a priori information and insures that certain constraints are satisfied. This is reformulated into a mathematical programming problem. A hierarchical scheme of finding the optimal strategies is then obtained.

In Chapter 4, the approach suggested in Chapter 3 is used to find an off-line decomposition for the linear-quadratic-Gaussian problem. Both the lower level problems and the higher level problem can be solved explicitly. The optimal local control strategy for the i^{th} controller is found to consist of two parts: a closed

loop part depending on the difference of his a posteriori local state estimate and his a priori local state estimate and an open loop part depending on the coordinating parameters transmitted by the coordinator.

In Chapter 5, the on-line decomposition of stochastic dynamic systems when the coordinator collects measurements from the lower level controllers and sends out coordinating parameters periodically is considered. Both open loop feedback optimal coordination and closed loop optimal coordination is discussed. For open loop feedback optimal coordination, the results in Chapters 3 and 4 are used to treat the nonlinear and linear-quadratic-Gaussian cases. For closed loop optimal coordination a functional equation which has to be solved is derived. With the help of the solution of a special dynamic team we arrive at explicit solutions for the linear-quadratic-Gaussian case. This is compared with the corresponding solution from open loop feedback optimal coordination.

In Chapter 6, we review the philosophy of this thesis and summarize the results obtained. Suggestions for future research are also given.

5. Contribution of this Thesis

The main contribution of this Thesis is the simultaneous treatment of the issues of computation and information in the control of large-scale systems. The concept of decentralized information is extended to include decentralized a priori information as well as decentralized a posteriori information. Thus decentralized control schemes, both computational and informational, are obtained. To compensate for the loss in optimality, we introduce an extra coordinator who has the common a priori information and some a posteriori information and influences the lower level through coordinating parameters. This is a new approach to the control of large-scale systems.

For static systems, stochastic optimization problems which have both the features of team decision problems and resource allocation problems are considered. Again this approach considers decentralized computation and information simultaneously.

For dynamic systems, the distinction between the two types of periodic coordination is new. Specialization to the linear-quadratic-Gaussian case gives results which are intuitively attractive.

CHAPTER 2

DECOMPOSITION FOR A STATIC STOCHASTIC OPTIMIZATION PROBLEM

1. Introduction

In this chapter we consider the stochastic optimization problem of a static system consisting of several subsystems. Each subsystem has a decision agent which has noisy information on the state of the system. The overall objective of the system is the sum of individual objectives of the subsystems. The subsystems are uncoupled except for constraints, which couple them together. Contrary to the deterministic case, the constraints do not have to be satisfied exactly. Rather, the problem solver only requires the constraints to be satisfied on the average. We have thus a constrained stochastic optimization problem with several decision agents each having noisy and different information on the state. The many decision agent aspect of the problem has been considered under the heading of team theory [R1]. For a constrained deterministic problem with the special structure described above, a hierarchical decomposition has been obtained using mathematical programming [L1]. We shall consider the two aspects of the problem simultaneously and obtain a hierarchical decomposition. This static problem is not only interesting for its own sake but is also useful for the decomposition of dynamic systems.

In the next section we present an example to motivate the general problem that we will study in this chapter. In Section 3 we review some results in non-linear programming; these can be used to obtain the decomposition of a static optimization problem when the state of the system is observed exactly. In Section 4 the stochastic optimization problem is formulated for the case when the state of the system is not known

exactly. In Section 5 the decomposition of the stochastic problem is investigated. Conditions under which the decomposition is well-posed are given and related to the information structure of the system. In Section 6 these results are stated in terms of measurement functions. The stochastic version of the example is solved in Section 7 and compared with the deterministic solution.

2. An Example

Consider a manufacturing company with N divisions, each producing a set of different commodities using the same resources. The ith division produces $\underline{u}_i$ units of goods $\underline{G}_i$ from $\underline{A}_i\underline{u}_i$ units of raw material at a cost of $\underline{u}_i'\underline{R}_i\underline{u}_i$ where $\underline{R}_i$ is assumed to be a positive definite matrix.

The market price of $\underline{G}_i$ is $2\underline{\pi}_i$ and the total resources available are $\underline{v}$.

Given any price vector $2\underline{\pi}_i$ and production $\underline{u}_i$, the profit function of the ith division is

$$- f_i(\underline{u}_i,\underline{\pi}_i) = 2\underline{u}_i'\underline{\pi}_i - \underline{u}_i'\underline{R}_i\underline{u}_i \tag{2.2.1}$$

The total profit of the company is the sum of the profits of all the divisions, i.e.,

$$-f(\underline{u},\underline{\pi}) = - \sum_{i=1}^{N} f_i(\underline{u}_i,\underline{\pi}_i) \tag{2.2.2}$$

The objective of the company is to minimize the total loss (maximize the total profit) subject to the constraint that the total resources used are less than the total resources available. The problem is thus

Problem 2.1: $$\underset{\underline{u}_1,\ldots,\underline{u}_N}{\text{Minimize}} \sum_{i=1}^{N} \underline{u}_i'\underline{R}_i\underline{u}_i' - 2\underline{u}_i'\underline{\pi}_i \tag{2.2.3}$$

$$\sum_{i=1}^{N} \underline{A}_i\underline{u}_i - \underline{v} \leq \underline{0} \tag{2.2.4}$$

Remark: We could have imposed the additional constraint that $\underline{u}_i \geq \underline{0}$, $i=1,\ldots\ldots, N$ but for simplicity we have assumed implicitly that the $\underline{u}_i$'s would turn out to be non-negative when Problem 2.1 is solved.

In this example the state of the system consists of the price vector $\underline{\pi}_i$, i=1,...., N, the resource vector $\underline{v}$ and possibly the cost matrices $\underline{R}_i$ and the resource utilization matrices $\underline{A}_i$. The decisions to be chosen are $\underline{u}_i$, i=1,...., N. Calling the state as x we have the following general problem

<u>Problem 2.2:</u> Minimize $\sum_{i=1}^{N} f_i(u_i,x)$

$$\text{Subject to } \sum_{i=1}^{N} g_i(u_i,x) - g_o(x) \leq \underline{0} \tag{2.2.5}$$

For our example

$$f_i(u_i,x) = \underline{u}_i'\underline{R}_i\underline{u}_i - 2\underline{u}_i'\underline{\pi}_i \tag{2.2.6}$$

$$g_i(u_i,x) = \underline{A}_i\,\underline{u}_i \tag{2.2.7}$$

$$g_o(x) = \underline{v} \tag{2.2.8}$$

There are situations when the state of the system cannot be observed exactly, but is described probabilistically. Suppose now that $\underline{\pi}_i$ is measured by the ith division manager as

$$\underline{z}_i = \underline{C}_i\,\underline{\pi}_i + \underline{\theta}_i \qquad i=1,....,N \tag{2.2.9}$$

$\underline{v}$ is measured by the resource manager as

$$\underline{z}_0 = \underline{C}_0\,\underline{v} + \underline{\theta}_0 \tag{2.2.10}$$

$\underline{\pi}_i$, $\underline{\theta}_i$, i=1,.... N, $\underline{v}$ and $\underline{\theta}_0$ are random vectors independent of each other and having the normal distributions (assumed known)

$$E\{\underline{\pi}_i\} = \overline{\underline{\pi}}_i \quad ; \quad Var\{\underline{\pi}_i\} = \underline{\Pi}_i \tag{2.2.11}$$

$$E\{\underline{\theta}_i\} = \underline{0} \quad ; \quad Var\{\underline{\theta}_i\} = \underline{\Theta}_i \qquad i=1,\ldots,N \tag{2.2.12}$$

$$E\{\underline{v}\} = \bar{\underline{v}} \quad ; \quad Var\{\underline{v}\} = \underline{V} \tag{2.2.13}$$

$$E\{\underline{\theta}_0\} = \underline{0} \quad ; \quad Var\{\underline{\theta}_0\} = \underline{\Theta}_0 \tag{2.2.14}$$

All the information available are contained in the measurements $\underline{z}_i$, $i=0,\ldots,N$. The production of each division has to be based on his measurement and some other signal based on $\underline{z}_0$.

The objective of the company is to minimize the expected total loss. As for the resource constraint (2.2.4) it can no longer be satisfied exactly since $\underline{v}$ is not measured exactly. Instead, we require the total resources used to be less than the total resources available given the measurement $\underline{z}_0$, i.e.

$$E\left\{\sum_{i=1}^{N} \underline{A}_i \underline{u}_i - \underline{v} \mid \underline{z}_0\right\} \leq \underline{0} \tag{2.2.15}$$

The production of each division has to use some information contained in $\underline{z}_0$ because the resource constraint (2.2.15) has to be satisfied. We thus have the following problem,

<u>Problem 2.1A</u>: Minimize $E\left\{\sum_{i=1}^{N} \underline{u}_i'\underline{R}_i\underline{u}_i - 2\underline{u}_i'\underline{\pi}_i\right\}$ (2.2.16)

subject to

$$E\left\{\sum_{i=1}^{N} \underline{A}_i \underline{u}_i - \underline{v} \mid \underline{z}_0\right\} \leq \underline{0} \tag{2.2.17}$$

$$\underline{u}_i = \underline{\eta}_i(\underline{z}_i; \underline{z}_0) \qquad i=1,\ldots,N \tag{2.2.18}$$

<u>Remark</u>: $\underline{u}_i$ at most can depend on all the information contained in $\underline{z}_i, \underline{z}_0$. We shall show later that the optimal decision function in some cases can

be found in a hierarchical manner and operation of the company can be decentralized.

3. Decomposition of a Non-Linear Programming Problem

In this section, we present some results in non-linear programming. These give rise immediately to a decomposition method for deterministic problems. Later on they will be used to obtain a decomposition for the stochastic case.

Consider the mathematical programming problem.

Problem 2.3: Minimize $f(u_1, \ldots, u_N)$

$$\text{Subject to } g(u_1, \ldots, u_N) \leq \underline{0} \in R^p \tag{2.3.1}$$

$$u_i \in U_i \qquad i=1, \ldots, N$$

where

$$f(u_1, \ldots, u_N) = f_1(u_1) + \ldots\ldots + f_N(u_N) \tag{2.3.2}$$

$$g(u_1, \ldots, u_N) = g_1(u_1) + \ldots\ldots + g_N(u_N) - g_o \tag{2.3.3}$$

Except for the coupling constraint (2.3.1), the problem is essentially uncoupled. The constraint may be interpreted as the common resource available to all the decision makers. This structure has been exploited to give a hierarchical decomposition scheme for the solution of the problem using results in mathematical programming. We state one sufficient condition which makes this possible.

Theorem 2.3.1 (Saddle-point condition): Let f be a real-valued function defined on a subset C of a linear space U. Let g be a mapping from C into the Euclidean space R^p. Suppose there exists a $\underline{p}^* \in R^p$, $\underline{p}^* \geq \underline{0}$ and a $u^* \in C$ such that the Lagrangian $L(u,\underline{p}) \triangleq f(u) + \underline{p}'g(u)$ possesses a saddle-point at $u^*,\underline{p}^*$, i.e.,

$$L(u^*,\underline{p}) \leq L(u^*,\underline{p}^*) \leq L(u^*,\underline{p}^*) \tag{2.3.4}$$

for all $u \varepsilon C$, $\underline{p} \geq \underline{0}$ then u^* solves

$$\text{minimize } f(u)$$

$$\text{Subject to } g(u) \leq \underline{0} \qquad u \varepsilon C \tag{2.3.5}$$

The proof of this theorem is elementary [L2]. Note that there are no conditions on the convexity or differentiability of f or g. For equality constraints, the same result holds except that $\underline{p}$ is no longer required to be non-negative. The following theorem is due to Lasdon [L1].

Theorem 2.3.2: Suppose there exists a saddlepoint for the Lagrangian corresponding to Problem 2.3, then the following hierarchical scheme can be used to obtain a solution, provided the minimizing problem is well-posed.*

Lower level: $\quad$ Minimize $\tilde{L}_i(u_i,\underline{p}) = f_i(u_i) + \underline{p}' g_i(u_i)$

$$\text{Subject to } u_i \varepsilon U_i$$

$$i=1,\ldots.,N \tag{2.3.6}$$

Higher level: $\quad$ Maximize $\sum_{i=1}^{N} \tilde{L}_i^*(\underline{p}) - \underline{p}' g_0$

$$\text{Subject to} \quad \underline{p} \geq \underline{0} \tag{2.3.7}$$

where $\tilde{L}_i^*(\underline{p})$ is the minimum obtained in equation (2.3.6).

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*For some $\underline{p}$, the lower level problem may not have a solution. We thus have to limit $\underline{p}$ to the set $D = \{\underline{p} |$ the lower level problem has a solution $\}$.

Proof: We need the fact that the constrained saddle-point for $L(a,b)$, $a \in A$, $b \in B$ exists if and only if [vl]

$$\min_{a \in A} \max_{b \in B} L(a,b) = \max_{b \in B} \min_{a \in A} L(a,b) \tag{2.3.8}$$

The value of the saddle-point is also equal to either side of equation (2.3.8). Given any $\underline{p}$ we note that the minimization part on the right side of equation (2.3.8) can be split up into N minimization problems independent of each other. Specifically, we have

$$\max_{\underline{p} \geq \underline{0}} \min_{u} L(u,\underline{p}) = \max_{\underline{p} \geq \underline{0}} \min_{u} \left\{ \sum_{i=1}^{N} f_i(u_i) + \sum_{i=1}^{N} \underline{p}' g_i(u_i) - \underline{p}' g_o \right\}$$

$$= \max_{\underline{p} \geq \underline{0}} \sum_{i=1}^{N} \min_{u_i} \{ f_i(u_i) + \underline{p}' g_i(u_i) \} - \underline{p}' g_0 \tag{2.3.9}$$

Equations (2.3.6) and (2.3.7) are obtained by making the appropriate identifications. Q.E.D.

Theorem 2.3.2 suggests a way of finding the optimal $\underline{p}^*$ and u^* simultaneously. This requires giving $\tilde{L}_i^*(\underline{p})$ as a function of $\underline{p}$. There are numerical methods [Ll] by which the optimal solution is found recursively by choosing a new $\underline{p}_{t+1}$ depending on the result of optimizing the dual function $\sum_{i=1}^{N} \tilde{L}_i(u_i, \underline{p}_t)$. However, we are more interested in the structure of the decomposition, i.e., once an optimal $\underline{p}^*$ is found, the lower level problems are uncoupled. The significance of this is more obvious when we look at the parametric case given by Problem 2.2. For each x we have a

mathematical programming problem; x may be regarded as the state of the system which is known exactly. If we use the result of Theorem 2.3.2, the optimal $\underline{p}^*$ would be a function of x, i.e., $\underline{p}^*(x)$. With this optimal $\underline{p}^*(x)$, the lower level problems would be

$$\underset{u_i \in U_i}{\text{Minimize}}\ \tilde{L}_i(u_i, \underline{p}^*(x), x) = f_i(u_i, x) + \underline{p}^{*\prime}(x) g_i(u_i, x) \qquad i=1,\ldots., N \quad (2.3.11)$$

Thus we can regard the higher level and lower level decision makers as both making observations on the system. The higher level decision maker (coordinator) observes the state x, chooses the coordinating parameter $\underline{p}^*(x)$ and transmits it to the lower level. The lower level decision makers then use this, together with f_i and g_i and x to choose their optimal decisions. This is displayed in fig. 2.1.

Applying this result to the example given in Problem 2.1 we have the following decomposition:

<u>Lower level</u> (Division manager):

$$\underset{\underline{u}_i}{\text{Minimize}}\ \underline{u}_i' \underline{R}_i \underline{u}_i - 2\underline{u}_i' \underline{\pi}_i + \underline{p}' \underline{A}_i \underline{u}_i \qquad (2.3.12)$$

$$i=1,\ldots., N$$

Denote the optimal of (2.3.12) by $\tilde{L}_i^*(\underline{p})$

<u>Higher level</u> (Resource manager):

$$\underset{\underline{p} \geq \underline{0}}{\text{Max}}\ \sum_{i=1}^{N} \tilde{L}_i^*(\underline{p}) - \underline{p}'\underline{v} \qquad (2.3.13)$$

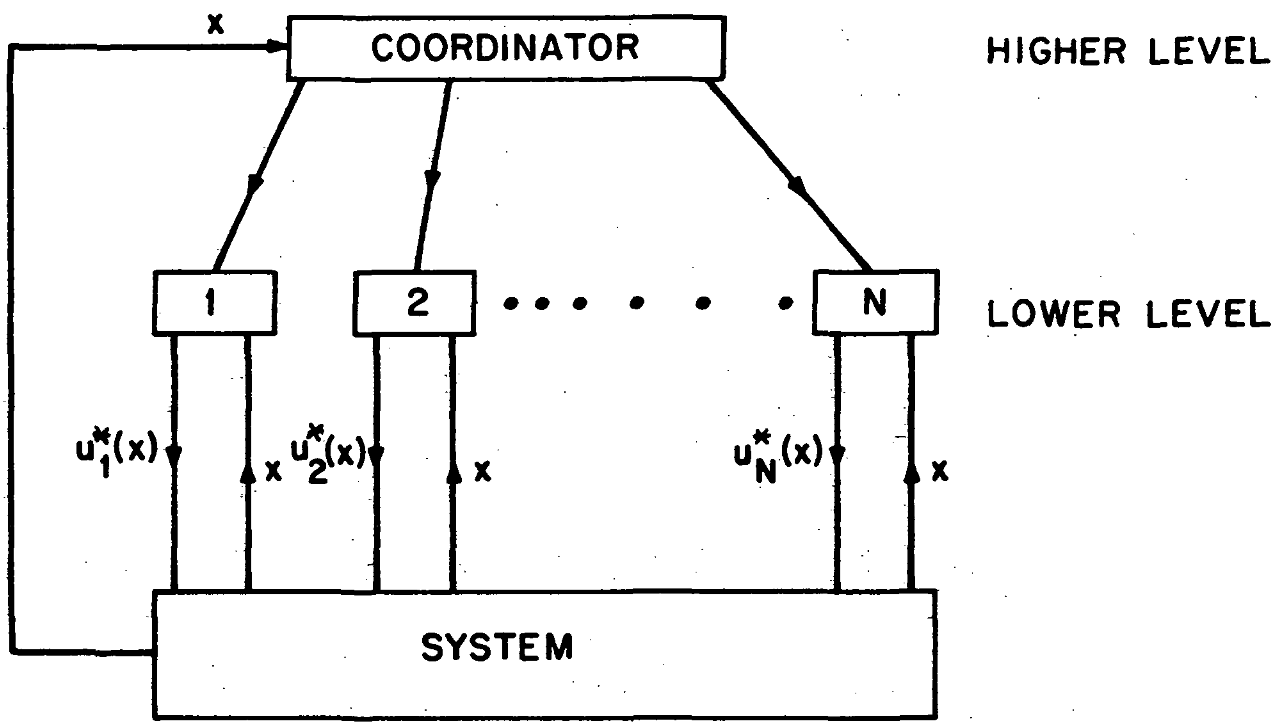

Fig. 2.1 Structure of Decomposition (Deterministic)

From these equations we obtain the following optimal $\underline{u}_i^*, i=1,\ldots, N$ and $\underline{p}^*$

$$\underline{u}_i^* = \underline{R}_i^{-1} [\underline{\pi}_i - \tfrac{1}{2}\underline{A}_i'\underline{p}^*] \tag{2.3.14}$$

$$\underline{p}^* = \underset{\underline{p} \geq \underline{0}}{\text{Arg Max}} - \tfrac{1}{4}\underline{p}'\left(\sum_{i=1}^{N} \underline{A}_i\underline{R}_i^{-1}\underline{A}_i'\right)\underline{p} + \underline{p}'\left(\sum_{i=1}^{N} \underline{A}_i\underline{R}_i^{-1}\underline{\pi}_i - \underline{v}\right) - \left(\sum_{i=1}^{N} \underline{\pi}_i'\underline{R}_i^{-1}\underline{\pi}_i\right) \tag{2.3.15}$$

Referring to equation (2.3.12) we see that the loss function of the ith division manager has been modified by the addition of an extra term which reflects the cost of resources. $\underline{p}$ is the price of the resources while $\underline{A}_i\underline{u}_i$ denotes the amount used.

In this deterministic case, the lower level decision makers base their decisions on $\underline{\pi}_i$ while the higher level bases his decision on $\underline{\pi}_i$ and $\underline{v}$. There is some decentralization of information, but the higher level in fact needs more information than the lower level. In the general deterministic case, both levels need the same information x, which is not too satisfactory. This leads us to study the stochastic case when information can also be decentralized.

4. Formulation of the Stochastic Problem

We now consider the case when the state x is not known exactly by the different decision makers. However, there is a probability description on the state space X given by the triplet $(X,\mathcal{B},\mu)$. $\mathcal{B}$ is a σ- algebra on X, and μ is a probability measure.

Let $\mathcal{F}_i, i=1,\ldots, N$ be sub-σ-algebras of $\mathcal{B}$. $\mathcal{F}_i$ represents the information available to the ith decision maker. Since the state x is not observed exactly, u_i will be required to be generated by a function γ_i measurable with respect to $\mathcal{F}_i$. This is equivalent to the existence of a measurement function h_i on x such that u_i depends on the measurement $z_i = h_i(x)$ [H1]. Denote by Γ_i the set of admissible decision functions γ_i measurable with respect to $\mathcal{F}_i$. Then $\gamma \triangleq (\gamma_1,\ldots,\gamma_N) \in \Gamma_1 \times \ldots \times \Gamma_N \triangleq \Gamma$. Given any decision function γ, $f(\gamma(x),x)$ would be a random variable. As in the case of team decision problems γ is chosen to minimize the expected payoff $E\{f(\gamma(x),x)\}$.

For the constraint several alternative formulations are possible.

1. $g(\gamma(x),x) \leq \underline{0}$ a.e. (2.4.1)

   As would be expected, it is rather difficult to satisfy this constraint.

2. $\text{Prob}\{g(\gamma(x),x) \leq \underline{0}\} \geq b$ (2.4.2)

   where b is some given probability.

Particular cases of this problem have been studied under the heading of <u>chance constrained programming</u> [C1]. It is the situation where the constraint is only required to be satisfied with a given probability.

3. $E\{g(\gamma(x),x)|F_0\} \leq 0$ a.e. (2.4.3)

where F_0 is some sub-σ-field of B. F_0 specifies the degree of exactness with which the constraint has to be satisfied or in other words the information of a coordinator who sees that the constraint is satisfied.

Two extreme cases are possible:

a. $F_0 = \{\Phi, X\}$ (2.4.4)

This corresponds to no measurements for the coordinator. Then

$$E\{g(\gamma(x),x)\} \leq 0 \tag{2.4.5}$$

b. $F_0 = B$ (2.4.6)

This corresponds to measuring the state almost exactly. Then

$$g(\gamma(x),x) \leq 0 \quad \text{a.e.} \tag{2.4.7}$$

With the introduction of the constraint, the information available to the decision makers may not be sufficient to insure that the constraint is satisfied. In general some extra information has to be communicated from the coordinator to the decision makers.

We will investigate what this information should be. Let $\Gamma_i' \supset \Gamma_i$ be the new admissible functions. Γ_i' is set of functions measurable with respect to $F_i \cap F_0$. Thus we have formulated the following stochastic optimization problem.

Problem 2.4: Minimize $E\{f(\gamma(x),x)\}$

Subject to $E\{g(\gamma(x),x)|F_0\} \leq 0$ a.e.

$$\gamma = (\gamma_1,\ldots,\gamma_N) \in \Gamma_1' x \ldots x \Gamma_N'$$

$$f(\gamma(x),x) = f_1(\gamma_1(x),x) + \ldots + f_N(\gamma_N(x),x)$$

$$g(\gamma(x),x) = g_1(\gamma_1(x),x) + \ldots + g_N(\gamma_N(x),x) - g_o(x) \qquad (2.4.8)$$

Remark: Γ_i' is the set of decision functions which use both the information of the ith decision maker as well as the information of the coordinator. We shall show later that not all the information of the coordinator is needed by the ith decision maker to choose his best decision. Under certain conditions, the information of the coordinator can be compressed into a signal which will be sufficient for the ith decision maker.

5. Decomposition of the Stochastic Problem

The special form of the constraint allows us to transform Problem 2.4 into a simpler form for which the results of section 3 are applicable.

Lemma 2.5.1: Let $f(\gamma(x),x)$ be a random function from $\Gamma' \times X$ into the reals, where Γ' is a set of functions on X measurable with respect to $F \cap F_0$. $F \subset B$ and $F_0 \subset B$. Γ is the set of functions measurable with respect to F.

Let $$M = \{\gamma | E\{g(\gamma(x),x)|F_0\} \leq \underline{0} \text{ a.e.}\}$$

Suppose $\underset{\gamma(\cdot;y) \in \Gamma \cap M}{\text{Min}} E\{f(\gamma(x;y),x)|F_0\}(y)$ exists a.e. and is equal

to $E\{f(\gamma^*(x;y),x)|F_0\}(y)$, then

$$\underset{\gamma \in \Gamma' \cap M}{\text{Min}} E\{f(\gamma(x),x)\} = E\{f(\gamma^*(x;x),x)\}$$
$$= E\{\underset{\gamma(\cdot;y) \in \Gamma \cap M}{\text{Min}} E\{f(\gamma(x;y),x)|F_0\}(y)\} \tag{2.5.1}$$

Proof: For $\gamma(\cdot) \in \Gamma' \cap M$ $\gamma(\cdot;y) \in \Gamma \cap M$

$$E\{f(\gamma(x;y),x)|F_0\}(y) = E\{f(\gamma(x),x)|F_0\}(y) \tag{2.5.2}$$

For a proof of this see Appendix A.

Thus

$$\underset{\gamma(\cdot;y) \in \Gamma \cap M}{\text{Min}} E\{f(\gamma(x;y),x)|F_0\}(y) = E\{f(\gamma^*(x;y),x)|F_0\}(y)$$
$$\leq E\{f(\gamma(x),x)|F_0\}(y) \text{ a.e. for all } \gamma \in \gamma' \cap M \tag{2.5.3}$$

Taking the unconditional expectation and minimizing over $\Gamma' \cap M$ we have,

$$E\{\min_{\gamma(\cdot;y)\,\varepsilon\,\Gamma\cap M} E\{f(\gamma(x;y),x)\,|\,F_0\}(y)\} \leq \min_{\gamma\,\varepsilon\,\Gamma'\cap M} E\{f(\gamma(x),x)\} \qquad (2.5.4)$$

On the other hand

$$E\{\min_{\gamma(\cdot;y)\,\varepsilon\,\Gamma\cap M} E\{f(\gamma(x;y),x)\,|\,F_0\}(y)\} = E\{f(\gamma^*(x),x)\} \geq \min_{\gamma\,\varepsilon\,\Gamma'\cap M} E\{f(\gamma(x),x)\} \qquad (2.5.5)$$

From equations (2.5.4) and (2.5.5) we obtain equation (2.5.1). Q.E.D.

Using Lemma 2.5.1, Problem 2.4 can be solved by considering the following problem.

Problem 2.5: Minimize $E\{f(\gamma(x;y),x)\,|\,F_0\}(y)$ a.e.

Subject to $E\{g(\gamma(x;y),x)\,|\,F_0\}(y) \leq 0$ a.e. (2.5.6)

$$\gamma(\cdot;y)\ \varepsilon\ \Gamma_1 x \ldots x \Gamma_N$$

If F_0 is such that the conditional probability measure $P_y{}^\circ(A)$ is regular, i.e. it is a probability measure given any y, then Problem 2.5 can be transformed to the following form.

Problem 2.6: Minimize $\hat{f}(\gamma;y)$

Subject to $\hat{g}(\gamma;y) \leq 0$

$$\gamma(\cdot;y)\ \varepsilon\ \Gamma \qquad (2.5.7)$$

where $\hat{f}(\gamma;y) \triangleq E\{f(\gamma(x;y),x)\,|\,F_0\}(y) = \int f(\gamma(x;y),x)\,dP_y{}^\circ(x)$ (2.5.8)

$$\hat{g}(\gamma;y) \triangleq E\{g(\gamma(x;y),x)\,|\,F_0\}(y) = \int f(\gamma(x;y),x)\,dP_y{}^\circ(x) \qquad (2.5.9)$$

Remark: The conditional probability measure is regular if it is generated by an observation function [D1].

Problem 2.6 is a conventional functional minimization problem given any y. The results in Theorems 2.3.1 and 2.3.2 do not depend on the finite

dimensionality of u, thus a hierarchical decomposition is obtained if a saddle-point exists for Problem 2.5. This is summarized in the following theorem.

Theorem 2.5.2: Suppose there exists a saddle-point $(\gamma^*(\cdot;y), p^*(y))$ for the Lagrangian associated with Problem 2.5. Then Problem 2.5 can be solved by the following hierarchical decomposition.

Lower level:

$$\text{Minimize } \tilde{L}_i(\gamma_i(\cdot;y), \underline{p}(y), y) = E\{f_i(\gamma_i(x;y),x) + \underline{p}'(y)g_i(\gamma_i(x;y),x) \,|\, F_0\}(y)$$

$$\gamma_i(\cdot;y) \in \Gamma_i \qquad i=1,\ldots\ldots, N \qquad (2.5.10)$$

Higher level:

$$\text{Maximize } \sum_{i=1}^{N} \tilde{L}_i^*(\underline{p}(y),y) - E\{\underline{p}'(y)g_0(x) \,|\, F_0\}(y)$$

$$\underline{p}(y) \geq 0 \qquad (2.5.11)$$

where $\tilde{L}_i^*(\underline{p}(y),y)$ is the minimum obtained in equation (2.5.10).

Proof: By using Theorem 2.3.2 on Problem 2.6, the decomposition is obtained.

Corresponding to Problem 2.4 we have the following decomposition.

Higher level: Choose $\underline{p}^*(y)$ measurable with respect to F_0.

Lower level:

$$\text{Minimize } \tilde{L}_i(\gamma_i(\cdot;y), \underline{p}^*(y), y) = E\{f_i(\gamma_i(x;y),x) + \underline{p}^{*\prime}(y)g_i(\gamma_i(x;y),x) \,|\, F_0\}(y)$$

$$\gamma_i(\cdot;y) \in \Gamma_i \qquad i=1.\ldots.., N \qquad (2.5.12)$$

Note the optimal γ_i^* can be expressed in the form $\gamma_i^*(x,p^*(x))$.

The optimization problem of each lower level decision maker is described by equation (2.5.12). A conditional expectation has to be optimized by each. <u>This optimization is not always well-defined with the information available to the ith decision maker</u>. We give a necessary and sufficient condition when this is defined.

<u>Theorem 2.5.3</u>: Let G_i be the smallest σ - algebra of F_0 with respect to which $E\{f_i(\gamma_i(x;y),x) + \underline{p}'(y)g_i(\gamma_i(x;y),x)|F_0\}$ is measurable. Then given $\underline{p}(y)$, $L_i(\gamma_i(\cdot;y),\underline{p}(y),y)$ can be optimized by the ith decision maker if and only if $G_i \subset F_i$.

<u>Proof</u>: For any measurable function $\ell(x)$, if $E\{\ell|F_0\}$ is measurable with respect to G_i, $G_i \subset F_0$, then $E\{\ell|F_0\} = E\{\ell|G_i\}$. (see Appendix A). If $G_i \subset F_i$, then

$$E\{f_i(\gamma_i(x;y),x) + \underline{p}'(y)g_i(\gamma_i(x;y),x)|F_0\}$$

$$= E\{f_i(\gamma_i(x;y),x) + \underline{p}'(y)g_i(\gamma_i(x;y),x)|G_i\}$$

$$= E\{E\{f_i(\gamma_i(x;y),x) + \underline{p}'(y)g_i(\gamma_i(x;y),x)|F_i\}|G_i\} \tag{2.5.13}$$

The inner expectation can be evaluated by the ith agent and minimized with respect to $\gamma_i(.;y) \in \Gamma_i$, hence minimizing $\tilde{L}_i(\gamma_i(\cdot;y),\underline{p}(y),y)$. If $G_i \not\subset F_i$, then $E\{f_i(\gamma_i(x;y),x) + \underline{p}'(y)g_i(\gamma_i(x;y),x)|G_i\}$ cannot be evaluated given the information contained in F_i, and thus it cannot be minimized. Q.E.D.

G_i represents the minimal sufficient information required by the ith agent to solve the decomposed decision problem given only $\underline{p}(y)$. If this information is not available, then the coordinator has to supply something else besides $\underline{p}(y)$. Typically this would be $P_y^{\,i}(A)$, the

conditional probability measure with respect to G_i. Note that although $F_0 \subset F_i$ satisfies the condition in Theorem 2.5.3, it is not always necessary for the ith agent to have more information than the coordinator. This will be illustrated in the next section.

6. Reformulation in Terms of Measurement Functions

In order to gain more insight, we shall reformulate the problem in terms of probability densities and measurement functions. The information requirements for the hierarchical decomposition can then be seen more easily.

Let x be the state of the system. x includes noises as well.

$z_i = h_i(x)$ be the measurement of the ith agent; $z_i \in Z_i$

$z_0 = h_0(x)$ be the measurement of the coordinator (specifying the constraint); $z_0 \in Z_0$

Then F_i, i=1,...., N is the σ - field on X generated by h_i and γ_i is measurable with respect to F_i if $\gamma_i = \eta_i o h_i$ where η_i is Borel-measurable on Z_i.

Corresponding to Problem 2.4 we have

<u>Problem 2.7</u>: Minimize $E\{f(\eta(z),x)\}$

$$\text{Subject to } E\{g(\eta(z),x)|z_0\} \leq \underline{0}$$

$$\eta(z) = (\eta_1(z_1;z_0),.....\eta_N(z_N;z_0))$$

$$f(\eta(z),x) = f_1(\eta_1(z_1;z_0),x) + ... + f_N(\eta_N(z_N;z_0),x)$$

$$g(\eta(z),x) = g_1(\eta_1(z_1;z_0),x) + ... + g_N(\eta_N(z_N;z_0),x) - g_0(x) \quad (2.6.1)$$

Corresponding to Problem 2.5, we have

<u>Problem 2.8</u>: Minimize $E\{f(\eta(z),x)|z_0\}$

$$\text{Subject to } E\{g(\eta(z),x)|z_0\} \leq \underline{0}$$

with η, f and g given as in equation (2.6.1) (2.6.2)

Theorem 2.5.2 then becomes

Theorem 2.6.1: Suppose there exists a saddle-point $(\eta^*(\cdot;z_0),\underline{p}^*(z_0))$ for the Lagrangian associated with Problem 2.8, then Problem 2.8 can be solved by the following hierarchical decomposition.

Lower level:

Minimize

$$\tilde{L}_i(\eta_i(\cdot;z_0),\underline{p}(z_0),z_0) = E\{f_i(\eta_i(z_i;z_0),x) + \underline{p}'(z_0)g_i(\eta_i(z_i;z_0),x)\,|\,z_0\}$$

$$i=1,\ldots\ldots, N \tag{2.6.3}$$

Higher level:

$$\text{Maximize} \sum_{i=1}^{N} \tilde{L}_i^*(\underline{p}(z_0),z_0) - E\{\underline{p}'(z_0)g_0(x)\,|\,z_0\}$$

$$\text{Subject to } \underline{p}(z_0) \geq \underline{0} \tag{2.6.4}$$

$\tilde{L}_i^*(\underline{p}(z_0),z_0)$ is the minimum obtained in equation (2.6.3).

Remark: From equation (2.6.3) we conclude that $\eta_i^*(z_i;z_0) = \eta_i^*(z_i;\underline{p}^*(z_0))$, i.e., all the relevant information about the constraint is contained in $\underline{p}^*(z_0)$ if the lower level problem is well defined.

The hierarchical decomposition scheme for Problem 2.7 then consists of the following.

Higher level: Coordinator makes a measurement z_0, computes the coordinating parameter $\underline{p}^*(z_0)$ and sends it to the lower level.

Lower level: ith decision agent makes a measurement z_i, and uses this together with $\underline{p}^*(z_0)$ to compute the best decision function $\eta_i^*(z_i;\underline{p}^*(z_0))$.

The structure of the decomposition is displayed in Figure 2.2. Note that the decomposition is in real-time since no iterations are involved.

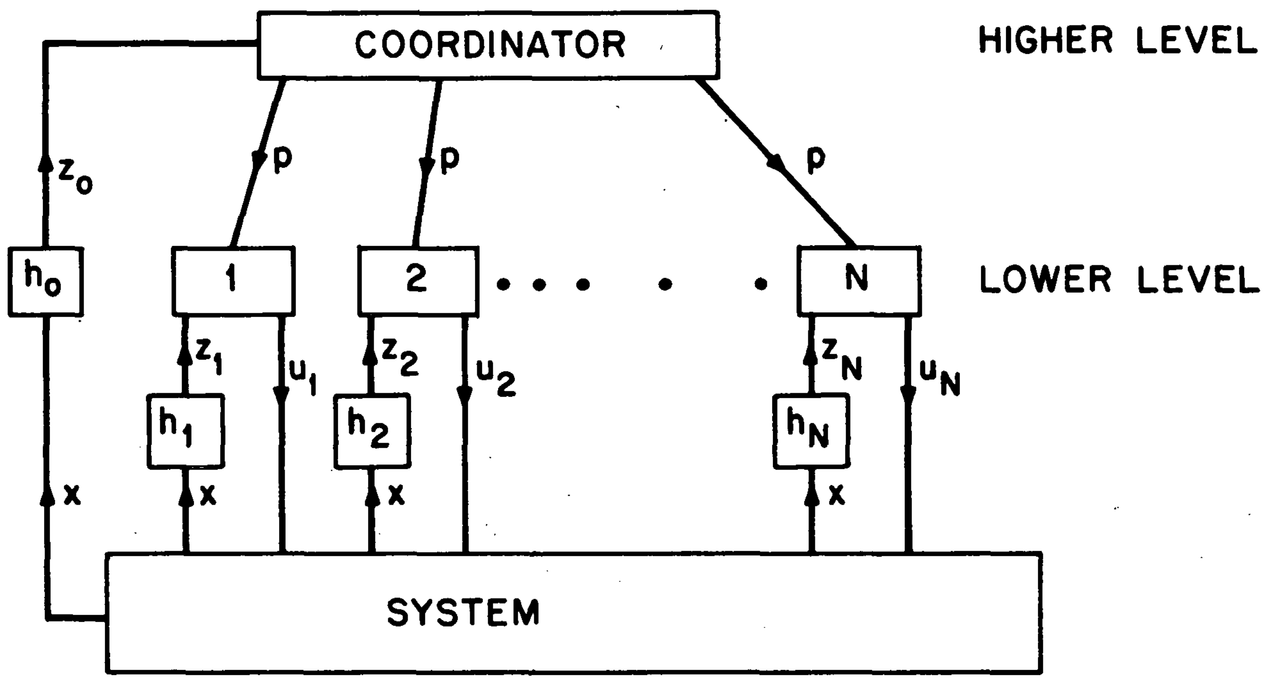

Fig. 2.2 Structure of Decomposition (Stochastic)

Because of the static nature of the problem, the information flow between the coordinator and lower level decision makers is only one-way.

An alternative condition for Theorem 2.5.3 is the following.

<u>Theorem 2.6.2</u>: $\tilde{L}_i(\eta_i(\cdot;z_0),\underline{p}(z_0),z_0)$ can be optimized by the ith decision maker if and only if

$$E\{f_i(\eta_i(z_i;z_0),x) + \underline{p}'(z_0)g_i(\eta_i(z_i;z_0),x)|z_i,z_0\}$$

$$= E\{f_i(\eta_i(z_i;z_0),x) + \underline{p}'(z_0)g_i(\eta_i(z_i;z_0),x)|z_i\} \qquad (2.6.5)$$

<u>Proof</u>: By the nested property of the conditional expectation [L4],

$$\tilde{L}_i(\eta_i(\cdot;z_0),p(z_0),z_0)$$

$$= E\{E\{f_i(\eta_i(z_i;z_0),x) + \underline{p}'(z_0)g_i(\eta_i(z_i;z_0),x)|z_i,z_0\}|z_0\} \qquad (2.6.6)$$

If the inner conditional expectation is equal to the right side of equation (2.6.5), then it can be minimized with respect to $\eta_i(\cdot;z_0)$. If equation (2.6.5) does not hold, then $\tilde{L}_i(\eta_i(.;z_0),\underline{p}(z_0),z_0)$ depends on the specific value of z_0 and thus cannot be minimized with respect to the function $\eta_i(\cdot;\underline{p}(z_0))$. Q.E.D.

We now give the results relating to the information between z_0 and z_i.

(1) $z_0 \subset z_i$. (Coordinator has less information than ith decision agent)

Then condition (2.6.5) is automatically satisfied.

Thus $\quad u_i^* = \eta_i^*(z_i;\underline{p}^*(z_0)) \qquad (2.6.7)$

(2) $z_0 \not\subset z_i$. (Coordinator has some information not available to ith decision agent).

(a) Condition (2.6.5) is satisfied $u_i^* = \eta_i^*(z_i;\underline{p}^*(z_0))$

Examples: (i) $f_i(x) = f_i(x_i) \quad g_i(x) = g_i(x_i)$ (2.6.8)

$$z_i = h_i(x_i) \qquad z_0 = h_0([x_i]) \tag{2.6.9}$$

where x_i and $[x_i]$ are statistically independent.

(ii) $f_i(x) = f_i(x_i) \quad g_i(x) = g_i(x_i)$ (2.6.10)

$$z_i = h_i(x_i) \qquad z_0 = \begin{bmatrix} h_0^1([x_i]) \\ h_0^2(x_i) \end{bmatrix} = \begin{bmatrix} z_0^1 \\ z_0^2 \end{bmatrix} \tag{2.6.11}$$

$$z_0^2 \subset z_i \tag{2.6.12}$$

(b) Condition (2.6.5) is violated.

$$\begin{aligned} u_i^* &= \eta_i^*(z_i;z_0) \\ &= \eta_i^*(z_i;P(x|z_0)) \end{aligned} \tag{2.6.13}$$

where $P(x|z_0)$ is the conditional probability density of x given z_0. In this case z_i and $\underline{p}^*(z_0)$ are no longer a sufficient statistics for the ith decision maker.

In words, if the coordinator has less information than the ith decision agent, as in the case when the information of the coordinator is shared by all decision agents, then the lower level problem is well defined given $\underline{p}(z_0)$ and the information of the ith decision agent. When this is not true, then the structure of the system and the information pattern has to be compatible in a certain sense, e.g. the state of the ith subsystem

is statistically independent from the rest of the system and the coordinator observes that state but this information is available to the ith decision agent.

Under other circumstances, the optimization problem for the ith decision agent may not be well-defined without the knowledge of z_0.

7. Solution of the Example

Using the results derived in the previous sections, the resource manager would charge an optimal price $\underline{p}^*(z_0)$ for the resources. Each division manager would then solve the following problem.

$$\underset{\underline{\eta}_i(\cdot;\underline{z}_0)}{\text{Minimize}}\ E\{\underline{\eta}_i'(\underline{z}_i;\underline{z}_0)\underline{R}_i\underline{\eta}_i(\underline{z}_i;\underline{z}_0) - 2\underline{\eta}_i'(\underline{z}_i;\underline{z}_0)\underline{\pi}_i + \underline{p}^{*\prime}(\underline{z}_0)\underline{A}_i\underline{\eta}_i(\underline{z}_i;\underline{z}_0) \mid \underline{z}_0\} \tag{2.7.1}$$

Since $\underline{\pi}_i$ is statistically independent of $\underline{v}$ and $\underline{\theta}_0$, the conditional expectation is equal to the unconditional expection given $\underline{p}^*(z_0)$. In fact the optimal $\underline{\eta}_i^*(\cdot;\underline{z}_0)$ is given by

$$\eta_i^*(\underline{z}_i;\underline{z}_0) = \underline{R}_i^{-1}\ [E\{\underline{\pi}_i \mid \underline{z}_i\} - \tfrac{1}{2}\underline{A}_i'\underline{p}^*(\underline{z}_0)\}] \tag{2.7.2}$$

The higher level problem is

$$\underset{\underline{p}(\underline{z}_0)\geq 0}{\text{Maximize}}\ \sum_{i=1}^{N} E\{\underline{\eta}_i^{*\prime}(\underline{z}_i^*;\underline{z}_0)\underline{R}_i\underline{\eta}_i^*(\underline{z}_i;\underline{z}_0) - 2\underline{\eta}_i^{*\prime}(\underline{z}_i;\underline{z}_0)\underline{\pi}_i + \underline{p}'(\underline{z}_0)\underline{A}_i\underline{\eta}_i^*(\underline{z}_i;\underline{z}_0) \mid \underline{z}_0\}$$

$$-E\{\underline{p}'(\underline{z}_0)\underline{v} \mid \underline{z}_0\} \tag{2.7.3}$$

$$E\{\underline{\eta}_i^{*\prime}(\underline{z}_i;\underline{z}_0)\underline{R}_i\underline{\eta}_i^*(\underline{z}_i;\underline{z}_0) - 2\underline{\eta}_i^{*\prime}(\underline{z}_i;\underline{z}_0)\underline{\pi}_i + \underline{p}'(\underline{z}_0)\underline{A}_i\underline{\eta}_i^*(\underline{z}_i;\underline{z}_0) \mid \underline{z}_0\}$$

$$= E\{ -\underline{\eta}_i^{*\prime}(\underline{z}_i;\underline{z}_0)\underline{R}_i\underline{\eta}_i^*(\underline{z}_i;\underline{z}_0) \mid \underline{z}_0\}$$

$$= -E\{(E\{\underline{\pi}_i \mid \underline{z}_i\} - \tfrac{1}{2}\underline{A}_i'\underline{p}(\underline{z}_0))^1\underline{R}_i^{-1}(E\{\underline{\pi}_i \mid \underline{z}_i\} - \tfrac{1}{2}\underline{A}_i'\ \underline{p}(\underline{z}_0)) \mid \underline{z}_0\}$$

$$= \frac{1}{4}\underline{p}'(\underline{z}_0)\underline{A}_i\underline{R}_i^{-1}\underline{A}_i'\underline{p}(\underline{z}_0) + \underline{p}'(\underline{z}_0)\underline{A}_i\underline{R}_i^{-1}\bar{\underline{\pi}}_i - c_i \tag{2.7.4}$$

$$c_i = -E\{E\{\underline{\pi}_i' \mid \underline{z}_i\}\underline{R}_i^{-1}E\{\underline{\pi}_i \mid \underline{z}_i\}\} = \text{constant} \tag{2.7.5}$$

Thus

$$\underline{p}^*(\underline{z}_0) = \operatorname*{Arg\ Max}_{\underline{p}(z_0) \geq 0} -\frac{1}{4}\underline{p}'(\underline{z}_0)\left(\sum_{i=1}^{N} \underline{A}_i \underline{R}_i^{-1} \underline{A}_i'\right)p(\underline{z}_0)$$

$$+ \underline{p}'(\underline{z}_0)\left(\sum_{i=1}^{N} \underline{A}_i R_i^{-1} \bar{\underline{\pi}}_i - E\{\underline{v}|\underline{z}_0\}\right) - \sum_{i=1}^{N} c_i \tag{2.7.6}$$

Comparing with the deterministic case in Section 3 we see that some kind of certainty equivalence (separation) theorem holds. The lower level devision managers choose their optimal productions by replacing the actual prices of their products with the best estimates given their measurements. However, whereas in the deterministic case the resource manager needs both $\underline{\pi}_i$, i=1,..., N and $\underline{v}$ to arrive at the optimal decision, resulting in essentially no decentralization in information, now it is only necessary to have information on $\underline{v}$.

8. Discussion and Perspectives

The decomposition achieved in mathematical programming for a class of systems with the general structure described in Section 2.3 is really with respect to computation. To study a possible decentralization in information we have formulated the stochastic version. It is found that under certain conditions a hierarchical decomposition for the problem is possible. The lower level decision makers need only to get certain signals from the higher level coordinator in addition to their information on the system. When these conditions are not satisfied, then in general the signals are not sufficient.

Radner and Groves [R2,G1] have considered a resource allocation problem similar to the one mentioned here. However, in their treatment there exists a resource manager who is in charge of allocating the resources directly. In our formulation, the resource manager serves only a coordinator. In the deterministic case, these two formulations become the same since the lack of an information pattern reduces the problem to the case of a single decision maker.

CHAPTER 3

DECOMPOSITION FOR NONLINEAR STOCHASTIC DYNAMIC SYSTEMS (OFF-LINE)

1. Introduction

In this chapter we consider the stochastic control of N coupled nonlinear subsystems. Each system has a controller who has noisy measurements on his subsystem. There is no communication between the controllers. The overall objective of the system consisting of all subsystems is the sum of individual objectives of the subsystems.

Because of the dynamic nature of the problem, the difficulties encountered here are different from those in static systems. Generally speaking, since the controls have to be applied in real time, on-line computation requirements for implementation of the optimal control strategy become important. The class of problems with different information patterns for the different controllers have been studied under the topic of dynamic teams [A1, C3, C4, H3]. So far, the results have not been very satisfactory in several respects. First, the optimal solution for even a linear-quadratic-Gaussian team is not known yet although there are indications of what the optimal solution should look like. Second, although the information structure in team decision problems is decentralized, often this is accompanied by an increase in both on-line and off-line computation. To give an example, let us consider the linear-quadratic-Gaussian problem. If information is centralized, then the optimal control strategy is given by the "separation" theorem and consists of the optimal deterministic control law acting on the estimate generated by the Kalman-Bucy filter [A2, M2, T1]. The on-line computation can be replaced by building a finite-dimensional filter.

However, if information is decentralized, then the on-line computation is extremely involved since each controller has to remember all his past observations or an "infinite-dimensional filter" is required. For a discussion of this, see Willman [W1]. As for the off-line computation, little is known since the optimal solution is not available. However, the computation involved in finding a suboptimal solution to the dynamic team problem has been shown to be relatively complicated [C3].

Since the computation and implementation of a control strategy is as important as the optimality resulting from the strategy itself, we will formulate in this chapter an optimization problem which is computionally more feasible as well as informationally efficient. The special coupled structure of the system and the form of the cost functional will be exploited. The concept of information structure is extended to include a priori information as well as a posteriori information. Thus the local controllers will not only have measurements on their subsystems alone, but will also be ignorant about the structure of the other subsystems. The coupled nature of the subsystems is taken care of by a coordinator who sees that certain constraints are satisfied. In this chapter we study the case when the coordinator has only a priori information, i.e. he does not make any measurements. In Chapter 5, we investigate the case when the coordinator makes on-line measurements.

The dynamic team problem is stated in the next section. A decomposition for the deterministic problem is then stated. This will be used to motivate the formulation of the stochastic decomposition problem in Section 3. In Section 4 we formulate a constrained stochastic optimal

control problem as a mathematical programming problem. In Section 5, the problem formulated in Section 3 is decomposed.

2. Statement of the Problem

We consider a discrete-time system consisting of N subsystems coupled together.

$$\underline{x}_i(k+1) = \underline{f}_i(\underline{x}_i(k), \underline{v}_i(k), \underline{u}_i(k), \xi_i(k)) \qquad i=1,\ldots, N \tag{3.2.1}$$

$$\underline{v}_i(k) = \sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(k)) \tag{3.2.2}$$

where

$\underline{x}_i(k) \in R^{n_i}$ is the "state" of the ith subsystem.

$\underline{v}_i(k) \in R^{q_i}$ is the action on the ith subsystem due to the other N-1 subsystems.

$\underline{u}_i(k) \in R^{p_i}$ is the control on the ith subsystem.

$\underline{\xi}_i(k) \in R^{r_i}$ is the driving noise on the ith subsystem.

$\underline{f}_i$ is the state transition function.

Let
$$\underline{x}(k) = \begin{bmatrix} \underline{x}_1(k) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \underline{x}_N(k) \end{bmatrix} \quad \underline{u}(k) = \begin{bmatrix} \underline{u}_1(k) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \underline{u}_N(k) \end{bmatrix} \quad \underline{\xi}(k) = \begin{bmatrix} \underline{\xi}_1(k) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \underline{\xi}_N(k) \end{bmatrix} \tag{3.2.3}$$

Then $\underline{v}_i(k)$, $i=1,\ldots, N$ can be eliminated from equation (3.2.1) to obtain a description for the whole system as

$$\underline{x}(k+1) = \underline{f}(\underline{x}(k), \underline{u}(k), \underline{\xi}(k)) \tag{3.2.4}$$

where the function $\underline{f}$ is defined in an obvious manner.

The description in terms of equations (3.2.1) and (3.2.2) is preferred here to display the coupled nature of the system. Note that even though $\underline{x}(k)$ can be regarded as the state of the system if the driving noise is absent, $\underline{x}_i(k)$ is, strictly speaking, not a state for the ith subsystem since knowledge of $\underline{x}_i(k)$, together with all the control $\underline{u}_i(j)$, $j \geq k$ is not sufficient to determine the future behavior of the ith subsystem.

The cost functional for the whole system is a sum of cost functionals for the individual subsystems, i.e.,

$$J = \sum_{i=1}^{N} J_i \tag{3.2.5}$$

$$J_i = E\{K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k), \underline{u}_i(k))\} \tag{3.2.6}$$

It is required to minimize J. The expectation is taken with respect to all the primitive random variables.

The problem is not yet well defined because we have not specified the information pattern of the system.

Let
$$\underline{y}_i(k) = \underline{h}_i(\underline{x}_i(k), \underline{\theta}_i(k)) \qquad i=1,\ldots\ldots, N \tag{3.2.7}$$

$\underline{y}_i(k) \in R^{m_i}$ is the measurement on the ith subsystem by the ith controller.

$\underline{\theta}_i(k) \in R^{m_i}$ is the noise corrupting the measurement.

Let
$$Y(k) = \{\underline{y}_i(s) \;\; 0 \leq s \leq k, \quad i=1,\ldots\ldots, N\} \tag{3.2.8}$$

$$U(k) = \{\underline{u}_i(s);\quad 0 \leq s \leq k,\quad i=1,\ldots\ldots, N\} \quad k=0,\ldots,T-1 \tag{3.2.9}$$

Let $(Y_i(k), U_i(k-1), I_i)$ be the information available to the ith controller at time k.

$$Y_i(k) \subset Y(k) \;;\; U_i(k-1) \subset U(k-1) \tag{3.2.10}$$

I_i is the a priori information of the entire system available to the ith controller.

Then $\underline{u}_i(k)$ is required to be a measurable function of $Y_i(k)$ and $U_i(k-1)$ which can be generated from I_i, i.e.,

$$\underline{u}_i(k) = \underline{\gamma}_i^{\,k}(Y_i(k),\, U_i(k-1);\, I_i) \tag{3.2.11}$$

I_i is introduced to take into consideration structural information of the system. The information available to the ith controller thus consists of two kinds: a priori (structual) information of the system and a posteriori (measurement) information. I_i essentially specifies the complexity of the control strategy. In the system given, if $I_i = \{\underline{f}_i, J_i, \underline{h}_i\}$, then as far as each controller is concerned, he is controlling an uncoupled system with an unknown input $\underline{v}_i(k)$. His control law $\underline{\gamma}_i^{\,k}$ would thus depend only on the parameters of his subsystem. This control law is thus "simpler", although a "loss" in mathematical optimality results. In most of the work done thus far, [H3, C4, C3] decentralization refers mainly to measurements, i.e., a posteriori information. The structure of the whole system is assumed known to each controller. With this a priori information, decentralized a posteriori information almost inevitably gives rise to a more complicated control strategy than centralized a posteriori information because each controller tries to generate the missing measurements using the common a priori information. The amount of on-line computation involved always increases, as well as the amount of off-line

computation. Even when the on-line computation is constrained by choosing suboptimal control structures, as in Chong and Athans [C3], the off-line computation required is still tremendous. In the implementation of control laws, computation considerations are as important as information considerations. This leads us to consider decentralized a priori information, at a sacrifice in overall optimality.

There is some work in the control literature which is vaguely related to decentralized a priori information. This is found in Ref. [M1] and [P1] and can be essentially illustrated by the following theorem for deterministic systems.

Theorem 3.2.1: Consider the optimal control problem given by

System:
$$\underline{x}_i(k+1) = \underline{f}_i(\underline{x}_i(k), \underline{v}_i(k), \underline{u}_i(k)) \quad i=1,\ldots,N \tag{3.2.12}$$

$$\underline{v}_i(k) = \sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(k)) \qquad \underline{x}(0) \text{ given}$$

Cost functional:
$$J = \sum_{i=1}^{N} J_i$$

$$J_i = K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k), \underline{u}_i(k)) \tag{3.2.13}$$

Suppose there exists a constrained saddle-point $(\underline{x}^*, \underline{u}^*, \underline{v}^*, \underline{p}^*)$ to the problem

$$L(\underline{x}^*, \underline{u}^*, \underline{v}^*, \underline{p}) \leq L(\underline{x}^*, \underline{u}^*, \underline{v}^*, \underline{p}^*) \leq L(\underline{x}, \underline{u}, \underline{v}, \underline{p}^*) \tag{3.2.14}$$

where $\underline{x} = \{\underline{x}_i(k); \quad k=1,\ldots, T; \quad i=1,\ldots, N\}$

$\underline{u} = \{\underline{u}_i(k), \quad k=0,\ldots, T-1; \; i=1,\ldots, N\}$

$\underline{v} = \{\underline{v}_i(k); \quad k=0,\ldots, T-1; \; i=1,\ldots, N\}$

$\underline{p} = \{\underline{p}_i(k); \quad k=0,\ldots, T-1; \; i=1,\ldots, N\}$

$\underline{x},\underline{u},\underline{v}$ are constrained by equation (3.2.12)

$$L(\underline{x},\underline{u},\underline{v},\underline{p}) = J + \sum_{i=1}^{N} \sum_{k=0}^{T-1} \underline{p}_i'(k)(\underline{v}_i(k) - \sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(k))) \qquad (3.2.15)$$

Then the optimal control problem can be solved as a two-level problem.

<u>Lower Level</u>: Minimize $\underset{\underline{u}_i,\underline{v}_i}{} \tilde{J}_i(\underline{u}_i,\underline{v}_i,\underline{p})$

$$\tilde{J}_i(\underline{u}_i,\underline{v}_i,\underline{p}) = K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k),\underline{u}_i(k)) + \underline{p}_i'(k)\underline{v}_i(k)$$

$$- \sum_{j\neq i} \underline{p}_j'(k)\underline{g}_{ji}(\underline{x}_i(k)) \qquad (3.2.16)$$

$$\underline{x}_i(k+1) = \underline{f}_i(\underline{x}_i(k),\underline{v}_i(k),\underline{u}_i(k)) \qquad (3.2.17)$$

<u>Higher Level</u>: $\max_{\underline{p}} \sum_{i=1}^{N} \tilde{J}_i^*(\underline{p}) \qquad (3.2.18)$

where $\tilde{J}_i{}^*(\underline{p})$ is the minimum obtained in equation (3.2.16).

<u>Proof</u>: The results in Section 2.3 are used. $L(\underline{x},\underline{u},\underline{v},\underline{p})$ is split up into uncoupled $\tilde{J}_i$'s by collecting all the terms involving $\underline{x}_i,\underline{v}_i$ and $\underline{u}_i$.

If the optimal $\underline{p}^*$ is given, then the lower level control problems are all uncoupled. The optimal control $\underline{u}^*_i$ can be found using only the structure of the ith system (its system dynamics and cost functional) plus the interconnection functions $g_{ji}(\cdot)$, $j\neq i$. The structural information of the other subsystems are not required. On the other hand, $\underline{p}^*$ is determined using all the optimal $\tilde{J}_i{}^*(\underline{p})$'s. Although algorithms can be devised making use of the special two-level structure of the optimization problem, the convergence to the optimal solution is not accomplished in real time [P1]. Thus the decomposition achieved is really with respect to the

off-line computation. In the deterministic problem given above, this corresponds to finding the open-loop control functions in some decentralized manner. In the next section we shall show that this philosophy can be extended to the stochastic case.

3. Formulation of the Stochastic Decomposition Problem

In the deterministic case given above, $\underline{v}_i$ is the action of the other subsystems on the ith subsystem, a quantity which is needed for the optimal control of the ith subsystem but is not itself optimized.

However, if the constraint $\underline{v}_i(k) = \sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(k))$ is satisfied exactly, optimizing with respect to $\underline{u}_i$ and $\underline{v}_i$ simultaneously is equivalent to solving the original optimal control problem with $\underline{u}_i$ as the only control to be optimized. In the actual implementation of the control, only $\underline{u}_i$ is used.

For each lower level problem, $\underline{v}_i$ can be regarded as an estimate of the interaction given $\underline{p}$. If the optimal $\underline{p}^*$ is used, then $\underline{v}_i$ is equal to the action of the other subsystems exactly.

We now extend this philosophy to the stochastic case. Instead of solving for the problem described by equations (3.2.1), (3.2.2), (3.2.5), (3.2.6) and (3.2.11) we shall exploit the coupled nature of the system. Since $\underline{x}(k)$ given the control strategy is a random vector, it is no longer possible to choose $\underline{v}_i(k)$ such that it equals $\sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(k))$ exactly. Rather $\underline{v}_i(k)$ is only required to be an estimate of the interaction and this is the job of the coordinator. We thus have the following formulation.

Problem 3.1:

Given
$$\underline{x}_i(k+1) = \underline{f}_i(\underline{x}_i(k), \underline{v}_i(k), \underline{u}_i(k), \underline{\xi}_i(k)) \qquad (3.3.1)$$
$$i=1,\ldots, N$$

$$J = \sum_{i=1}^{N} J_i$$

$$J_i = E\{K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k), \underline{u}_i(k))\} \tag{3.3.2}$$

$$E\{\underline{v}_i(k) - \sum_{j \neq i}^{N} \underline{g}_{ij}(\underline{x}_j(k))\} = 0 \qquad i=1,\ldots., N \quad k=0,\ldots.T-1 \tag{3.3.3}$$

$$\underline{u}_i(k) = \underline{\gamma}_i^{\,k}(Y_i(k), U_i(k-1); \tilde{I}) \tag{3.3.4}$$

$$\underline{v}_i(k) = \underline{\eta}_i^{\,k}(\tilde{I}) \tag{3.3.5}$$

$$Y_i(k) = \{\underline{y}_i(s); 0 \leq s \leq k\} \tag{3.3.6}$$

$$U_i(k) = \{\underline{u}_i(s); 0 \leq s \leq k\} \tag{3.3.7}$$

Find $\underline{\gamma}_i^{\,k}$ and $\underline{\eta}_i^{\,h}$, i=1,...., N; k=0,....T-1 such that J is minimized. $\tilde{I}$ consists of the a priori information contained in the model and the cost functional.

The original stochastic control problem has been modified in the following manner. The subsystems are all assumed to be uncoupled. The interaction of the other subsystems is represented by $\underline{v}_i(k)$ which is to be optimized. $\underline{v}_i(k)$ is chosen, however, so that constraint (3.3.3) is satisfied; thus it is an unbiased a priori estimate of the interaction of the other subsystems. The control problem then consists of finding the optimal control strategies $\underline{\gamma}$ and the optimal estimates of the interactions such that the cost functional is minimized.

Although this problem is very similar to the deterministic problem given in Section 2 of this chapter, the results of decomposition in mathematical programming cannot be applied directly since closed loop control

strategies $\underline{\gamma}_i$ are required. In the next section, we show how the stochastic control problem can be reformulated so as to lead to a constrained optimization problem.

4. A Constrained Stochastic Optimal Control Problem

Consider the following stochastic control problem.

Problem 3.2:

System: $\underline{x}(k+1) = \underline{f}(\underline{x}(k),\underline{u}(k),\underline{\xi}(k)) \quad \underline{x}(k) \in R^n$ (3.4.1)

Measurement: $y(k) = \underline{h}(\underline{x}(k),\underline{\theta}(k)) \quad \underline{u}(k) \in R^p$ (3.4.2)

Cost functional: $J = E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k),\underline{u}(k))\}$ (3.4.3)

$\underline{\xi}(k),\underline{\theta}(k)$, $k=0,\dots, T-1$ and $\underline{x}(0)$ are random vectors with known statistics.

$$Y(k) \subseteq \{\underline{y}(0),\dots,\underline{y}(k);\ \underline{u}(0),\dots,\underline{u}(k-1)\} \tag{3.4.4}$$

$\underline{u}(k)$ is constrained to be an admissible function of $Y(k)$, i.e.,

$$\underline{u}(k) = \gamma^k(Y(k)) \qquad \gamma \in \Gamma \tag{3.4.5}$$

It is required to choose $\gamma^* \in \Gamma$ such that

$$J(\gamma^*) = \min_{\gamma \in \Gamma} J(\gamma) \tag{3.4.6}$$

In the problem stated above, the minimization is only over the strategy space Γ. We can transform this to a minimization over random sequences subject to certain constraints.

Let the underlying probability spaces be $(\Omega,\mathcal{B},\mu)$. $\underline{\theta}(k)$, $\underline{\xi}(k)$, $\underline{x}(0)$ are random vectors over Ω.

Let $\underline{x}(\omega) = (\underline{x}(1,\omega),\dots,\underline{x}(T,\omega))$ be a $\mathcal{B}$-measurable L^2 function over Ω into R^{nT}, i.e., $\underline{x} \in L^2(\Omega, R^{nT})$

Let $\underline{u}(\omega) = (\underline{u}(0,\omega),\dots,\underline{u}(T-1,\omega)) \in L^2(\Omega,R^{pT})$.

Let

$$S_1 = \{\underline{x} \in L^2(\Omega,T^{nT}),\ \underline{u} \in L^2(\Omega,R^{pT}) \mid \underline{x}(k+1,\omega) = \underline{f}(\underline{x}(k,\omega),\ \underline{u}(k,\omega)) \text{ a.e.}\}$$

= set of $\underline{x},\underline{u}$ which correspond to the given dynamic system (3.4.7)

$$S_2 = \{\underline{x} \in L^2(\Omega,R^{nT}),\ \underline{u} \in L^2(\Omega,R^{pT}) \mid \exists\, \gamma \in \Gamma \text{ such that}$$

$$\underline{u}(k,\omega) = \gamma^k(Y(k,\omega)) \quad \text{a.e.}\}$$

$$= \{\underline{x} \in L^2(\Omega,R^{nT}),\ \underline{u} \in L^2(\Omega,R^{pT}) \mid \exists\, \gamma \in \Gamma \text{ such that}$$

$$\underline{u}(k,\omega) = \gamma^k(\underline{h}(\underline{x}(0,\omega),\ \underline{\theta}(0,\omega)),\ \underline{h}(\underline{x}(1,\omega),\ \underline{\theta}(1,\omega)),\ldots,$$

$$\underline{h}(\underline{x}(k,\omega),\ \underline{\theta}(k,\omega));\ \underline{u}(0,\omega),\ldots,\underline{u}(k-1,\omega)) \text{ a.e.}\}$$

= set of $\underline{x},\underline{u}$ which can be generated from the given information structure and admissible control strategy. (3.4.8)

Let $G : \Gamma \to L^2(\Omega,R^{nT}) \times L^2(\Omega,R^{pT})$ be defined as

$$G(\gamma) = (\underline{x}(\gamma),\ \underline{u}(\gamma)) \tag{3.4.9}$$

Then by the definition of S_1 and S_2,

$$\text{Range } G = S_1 \cap S_2 \tag{3.4.10}$$

Therefore

$$\begin{aligned} \min_{\gamma \in \Gamma} J(\gamma) &= \min_{\gamma \in \Gamma} J(\underline{x}(\gamma),\ \underline{u}(\gamma)) \\ &= \min_{G(\gamma) \in S_1 \cap S_2} J(\underline{x}(\gamma),\ \underline{u}(\gamma)) \\ &= \min_{(\underline{x},\underline{u}) \in S_1 \cap S_2} J(\underline{x},\underline{u}) \end{aligned} \tag{3.4.11}$$

Note that the minimization is now over random sequence $\underline{x},\underline{u}$. The dynamics of the system, the constraint on the control strategy and the

information structure allowed have been incorporated into the constraint set $S_1 \cap S_2$.

We next consider the constrained stochastic control problem.

Problem 3.3:

System: $$\underline{x}(k+1) = \underline{f}(\underline{x}(k), \underline{u}(k), \underline{\xi}(k)) \tag{3.4.12}$$

Measurement: $$\underline{y}(k) = \underline{h}(\underline{x}(k), \underline{\theta}(k)) \tag{3.4.13}$$

Cost Functional: $$J = E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k), \underline{u}(k))\} \tag{3.4.14}$$

$$\underline{u}(k) = \gamma^k(Y(k)) \tag{3.4.15}$$

$$E\{\underline{H}(\underline{x}(k), \underline{u}(k))\} = \underline{0} \in R^q \quad k = 0,\ldots T-1 \tag{3.4.16}$$

It is required to choose $\gamma^* \in \Gamma$ such that

$$J(\gamma^*) = \underset{\gamma \in \Gamma}{\text{Min}} \; J(\gamma)$$

and the constraint (3.4.16) is satisfied. $\underline{H}$ is a vector-valued function.

This constraint is only required to be satisfied on the average. Problem 3.3 can be transformed into the following unconstrained stochastic control problem.

Problem 3.4:

System: $$\underline{x}(k+1) = \underline{f}(\underline{x}(k), \underline{u}(k), \underline{\xi}(k)) \tag{3.4.17}$$

Measurement: $$\underline{y}(k) = \underline{h}(\underline{x}(k), \underline{\theta}(k)) \tag{3.4.18}$$

Cost Functional: $$\tilde{J}(\gamma,\underline{p}) = E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k), \underline{u}(k)) + \underline{p}'(k)\underline{H}(\underline{x}(k), \underline{u}(k))\} \tag{3.4.19}$$

$$\underline{u}(k) = \gamma^*(Y(k)) \tag{3.4.20}$$

It is required to find γ^* such that $\tilde{J}(\gamma,\underline{p})$ is minimized.

Theorem 3.4.1: Suppose a saddle point exists for the stochastic control problem 3.4, i.e. there exist $\gamma^*,\underline{p}^*$ such that

$$\tilde{J}(\gamma^*,\underline{p}) \le \tilde{J}(\gamma^*,\underline{p}^*) \le \tilde{J}(\gamma,\underline{p}^*) \tag{3.4.21}$$

Then γ^* is the solution to Problem 3.3.

Proof: The constraint (3.4.16) can be written as

$$\tilde{H}(\underline{x},\underline{u}) = \underline{0} \in R^{qT} \tag{3.4.22}$$

where

$$\underline{x} \in L^2(\Omega,R^{nT}),\ \underline{u} \in L^2(\Omega,R^{pT}) .$$

Problem 3.3 is then equivalent to

$$\underset{\underline{x},\underline{u} \in S_1 \cap S_2}{\text{Min}} \quad J(\underline{x},\underline{u})$$

$$\tilde{H}(\underline{x},\underline{u}) = \underline{0} \tag{3.4.23}$$

$$\begin{aligned}
\tilde{J}(\gamma,\underline{p}) &= E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k),\ \underline{u}(k)) + \underline{p}'(k)\underline{H}(\underline{x}(k),\ \underline{u}(k))\} \\
&= E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k),\ \underline{u}(k))\} + \sum_{k=0}^{T-1} \underline{p}'(k)\ E\{\underline{H}(\underline{x}(k),\ \underline{u}(k))\} \\
&= J(\gamma) + \underline{p}'\ \tilde{H}(\underline{x}(\gamma),\ \underline{u}(\gamma))
\end{aligned} \tag{3.4.24}$$

If $\tilde{J}(\gamma,\underline{p})$ has a saddle point $(\gamma^*,\underline{p}^*)$, then $(G(\underline{\gamma}^*),\underline{p}^*)$ is a saddle point for the function $J(\underline{x},\underline{u}) + \underline{p}'\underline{H}(\underline{x},\underline{u})$.

By Theorem 2.3.1, $G(\gamma^*) = (\underline{x}^*,\underline{u}^*)$ solves

$$\underset{(\underline{x},\underline{u}) \in S_1 \cap S_2}{\text{Min}} J(\underline{x},\underline{u})$$

such that

$$\tilde{H}(\underline{x},\underline{u}) = \underline{0} \tag{3.4.25}$$

or γ^* solves Problem 3.3. Q.E.D.

The following corollary follows immediately.

Corollary 3.4.2: If a saddle point $(\gamma^*,\underline{p}^*)$ exists for Problem 3.4, then the optimal strategy γ^* can be found by

$$\underset{\underline{p}}{\text{Max}}\ \underset{\gamma}{\text{Min}}\ E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k),\ \underline{u}(k)) + \underline{p}'(k)\underline{H}(\underline{x}(k),\ \underline{u}(k))\} \tag{3.4.26}$$

Proof: We need only the fact that if a saddle point $(\gamma^*,\underline{p}^*)$ exists for the function $L(\gamma,\underline{p})$, then

$$\underset{\gamma}{\text{Min}}\ \underset{\underline{p}}{\text{Max}}\ L(\gamma,\underline{p}) = \underset{\underline{p}}{\text{Max}}\ \underset{\gamma}{\text{Min}}\ L(\gamma,\underline{p}) = L(\gamma^*,\underline{p}^*) \tag{3.4.27}$$

To check for the saddle point, we need to verify the condition directly or use condition (3.4.27). The following condition is sometimes more convenient.

Lemma 3.4.3: Consider the problem

$$\underset{x}{\text{Min}}\ f(x)$$

$$g(x) = \underline{0} \quad x \in C \tag{3.4.28}$$

If

(1) $\underset{\underline{p}}{\text{Max}}\ \underset{x \in C}{\text{Min}}\ f(x) + \underline{p}'g(x) \triangleq f(x^*) + \underline{p}^{*\prime}g(x^*)$ exists

(2) $g(x^*) = \underline{0}$

then x^* minimizes $f(x)$ such that $g(x) = \underline{0}, x \in C$.

Proof:

$$f(x^*) + \underline{p}'g(x^*) = f(x^*) = f(x^*) + \underline{p}^{*\prime}g(x^*) \tag{3.4.29}$$

$$f(x) + \underline{p}^{*\prime} g(x) \geq \min_{x \in C} f(x) + \underline{p}^{*\prime}g(x) \tag{3.4.30}$$

$$\min_{x \in C} f(x) + \underline{p}^{*\prime} g(x) = f(x^*(\underline{p}^*)) + \underline{p}^{*\prime} g(\underline{x}^*(\underline{p}^*)) \tag{3.4.31}$$

where $x^*(\underline{p})$ minimizes $f(x) + \underline{p}'g(x)$, $x \in C$.

Thus

$$\max_{\underline{p}} \min_{x \in C} f(x) + \underline{p}'g(x) = \max_{\underline{p}} f(x^*(\underline{p})) + \underline{p}'g(x^*(\underline{p}))$$

$$= f(x^*(\underline{p}^*)) + \underline{p}^{*\prime}g(\underline{x}^*(\underline{p}^*)) \text{ by definition}$$

$$= \min_{x \in C} f(x) + \underline{p}^{*\prime}g(x) \tag{3.4.32}$$

Then

$$f(x^*) + \underline{p}'g(x^*) \leq f(x^*) + \underline{p}^{*\prime}g(x^*) \leq f(x) + \underline{p}^{*\prime}g(x)$$

$$\text{for all } x \in C \text{ and } \underline{p} \tag{3.4.33}$$

$(x^*, \underline{p}^*)$ is a saddle point and x^* minimizes $f(x)$ such that $g(x) = \underline{0}, x \in C$. Q.E.D.

Theorem 3.4.1 can then be restated in the following form.

Theorem 3.4.4: Suppose

$$\max_{\underline{p}} \min_{\gamma} E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k), \underline{u}(k)) + \underline{p}'(k)\underline{H}(\underline{x}(k), \underline{u}(\underline{k}))\}$$

exists for the system described in Problem 3.4, and further

$$E\{\underline{H}(\underline{x}^*(k),\ \underline{u}^*(k))\} = \underline{0} \qquad k = 0,\ldots,T-1$$

where $\underline{x}^*,\underline{u}^*$ are the optimal trajectory and control using γ^*. Then γ^* is the optimal strategy for Problem 3.3.

5. Decomposition of the Stochastic Control Problem

We now apply the results of the last section to Problem 3.1 and transform it to an unconstrained problem.

Theorem 3.5.1: Consider the system

$$\underline{x}_i(k+1) = \underline{f}_i(\underline{x}_i(k),\ \underline{v}_i(k),\ \underline{u}_i(k),\ \underline{\xi}_i(k)) \quad i=1,\dots,N \tag{3.5.1}$$

$$\underline{u}_i(k) = \underline{\gamma}_i^{\,k}(Y_i(k),\ U_i(k-1);\tilde{I}) \tag{3.5.2}$$

$$\underline{v}_i(k) = \underline{\eta}_i^{\,k}(\tilde{I}) \tag{3.5.3}$$

$$\tilde{J} = \sum_{i=1}^{N} \tilde{J}_i \tag{3.5.4}$$

$$\tilde{J}_i = E\{K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k),\ \underline{u}_i(k)) + \underline{p}_i'(k)\ \underline{v}_i(k) - \sum_{j\neq i} \underline{p}_j'(k)\ \underline{g}_{ji}(\underline{x}_i(k))\} \tag{3.5.5}$$

If $\underset{\underline{p}}{\text{Max}}\ \underset{\underline{\gamma},\underline{\eta}}{\text{Min}}\ \tilde{J}$ exists and

$$E\{\underline{v}_i^*(k) - \sum_{j\neq i}^{N} \underline{g}_{ij}(\underline{x}_j^*(k))\} = \underline{0} \quad i=1,\dots,N;\ \ k=0,\dots,T-1 \tag{3.5.6}$$

then $\underline{\gamma}^*,\underline{\eta}^*$ are the optimal strategies for Problem 3.1.

Proof: This problem can be cast into the form of Problem 3.3 by identifying

$$\underline{u}(k) \quad \text{with} \quad \{\underline{u}_i(k),\ \underline{v}_i(k);\ i=1,\dots,N\} \tag{3.5.7}$$

$$\gamma^k \quad \text{with} \quad \{\underline{\gamma}_i^{\,k},\ \underline{\eta}_i^{\,k};\ \ i=1,\dots,N\} \tag{3.5.8}$$

$$\underline{H}(\underline{x}(k),\ \underline{u}(k)) = \begin{bmatrix} \underline{v}_1(k) - \sum_{j\neq 1} g_{1j}(\underline{x}_j(k)) \\ \vdots \\ \underline{v}_N(k) - \sum_{j\neq N} g_{Nj}(\underline{x}_j(k)) \end{bmatrix} \tag{3.5.9}$$

$$\sum_{i=1}^{N} E\{K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k), \underline{u}_i(k)) + \underline{p}_i'(k)(\underline{v}_i(k) - \sum_{j\neq i}^{N} \underline{g}_{ij}(\underline{x}_j(k))\}$$

$$= \sum_{i=1}^{N} E\{K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k), \underline{u}_i(k)) + \underline{p}_i'(k)\ \underline{v}_i(k)$$

$$- \sum_{j\neq i} \underline{p}_j'(k)\ \underline{g}_{ji}(\underline{x}_i(k))\}$$

$$= \sum_{i=1}^{N} \tilde{J}_i \tag{3.5.10}$$

Theorem 3.4.4 can then be applied in a straight forward manner. Q.E.D.

Note that given any $\underline{p}$, the minimization problem is separated into N uncoupled stochastic control problems. The ith controller needs only the structure of his own system as his a priori information. Thus there is decentralization of a priori as well as a posteriori information.

A two-level hierarchical decomposition for finding the optimal control strategy is possible.

Lower Level:

$$\underline{x}_i(k+1) = \underline{f}_i(\underline{x}_i(k), \underline{v}_i(k), \underline{u}_i(k), \underline{\xi}_i(k))$$

$$\underline{u}_i(k) = \underline{\gamma}_i^k(Y_i(k), U_i(k-1); \tilde{I})$$

$$\underline{v}_i(k) = \underline{\eta}_i^k(\tilde{I})$$

$$\tilde{J}_i(\underline{p}) = E\{K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k), \underline{u}_i(k)) + \underline{p}_i'(k)\ \underline{v}_i(k)$$

$$- \sum_{j\neq i} \underline{p}_j'(k)\ \underline{g}_{ji}(\underline{x}_i(k))\} \tag{3.5.11}$$

Find $\underline{\gamma}_i^k$ and $\underline{\eta}_i^k$ such that $\tilde{J}_i(\underline{p})$ is minimized, $i=1,\ldots,N$. Let $\tilde{J}_i^*(\underline{p})$ be the optimal cost associated with a particular $\underline{p}$.

<u>Higher Level</u>: $$\underset{\underline{p}}{\text{Maximize}} \sum_{i=1}^{N} \tilde{J}_i^*(\underline{p}) \qquad (3.5.12)$$

<u>Remark</u>: The higher level problem is deterministic and static in nature whereas the lower level problems are stochastic and dynamic, although uncoupled. The decentralized a priori information allows the off-line computation to be done in an algorithmic manner. Typically, the higher level coordinator will choose a $\underline{p}$, the lower level controllers then compute the optimal cost associated with this $\underline{p}$. The coordinator then chooses another $\underline{p}$ to increase the optimal cost of the lower level systems. The decomposition is <u>off-line</u> because it is done before the system starts using only a priori information. The advantages of this approach are the following:

(1) The overall stochastic control problem is split up into N stochastic control problems with lower dimension. Each of these can be solved more easily.

(2) Although the value of $\underline{p}$ may change, the structure of the lower level problems remains the same, and hence essentially the stochastic control problems need only to be solved once.

6. Discussion and Perspectives

We have considered the stochastic control of N coupled systems with decentralized information structure. By defining a new kind of optimality, it is found that the optimal control strategies can be found in a decentralized manner. Moreover, given the optimal coordinating parameters, the control problems of the N subsystems are uncoupled. Thus the control strategies using decentralized a posteriori information can be computed with decentralized a priori information. Although this scheme is suboptimal with respect to the ordinary stochastic control problem, computationally it is more efficient.

Because of the nonlinear nature of the problem we cannot say much about the detailed computations involved. However, it is obvious that instead of one high dimensional stochastic control problem we now have N lower dimensional stochastic control problems and one extra deterministic optimization problem to be solved by the coordinator. In the next chapter, we shall look at the linear-quadratic-Gausian problem in detail and obtain explicit solutions for these lower and higher level problems.

CHAPTER 4

DECOMPOSITION FOR THE LINEAR-QUADRATIC-GAUSSIAN PROBLEM (OFF-LINE)

1. Introduction

In this chapter we apply the philosophy of Chapter 3 to the linear-quadratic-Gaussian problem. As pointed out in the introduction of Chapter 2, the solution to the linear-quadratic-Gaussian dynamic team is not known yet. Even if it is found, the on-line computation involved will make its implementation not feasible since the estimates involved have to be generated by infinite dimensional filters. The control strategies obtained in this chapter are easily implementable. The structure of the control decomposes very nicely into an open loop part and a closed loop part. This will be used later on to study the on-line "periodic" coordination of coupled systems (see Chapter 5).

In the next section, we formulate the LQG problem and decompose it into two levels. The equations needed by the lower level controllers and the coordinator are given in Section 3. The lower level problem with a linear term in the cost functional is solved in Section 4. In Section 5, the higher level problem is solved and found to bear a very close relationship to the deterministic linear quadratic problem.

2. Statement of the Problem

Consider the linear dynamic system

$$\underline{x}_i(k+1) = \underline{A}_{ii}\underline{x}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k) + \underline{\xi}_i(k) \quad i=1,\ldots,N \tag{4.2.1}$$

$$\underline{v}_i(k) = \sum_{j\neq i} \underline{A}_{ij}\underline{x}_j(k) \tag{4.2.2}$$

The cost functional is quadratic.

$$J = \sum_{i=1}^{N} J_i = \sum_{i=1}^{N} E\{\underline{x}_i'(T)\underline{F}_i\underline{x}_i(T) + \sum_{k=0}^{T-1} \underline{x}_i'(k)\underline{Q}_i\underline{x}_i(k) + \underline{u}_i'(k)\underline{R}_i\underline{u}_i(k)\} \tag{4.2.3}$$

where $\underline{F}_i$, $\underline{Q}_i$, $\underline{R}_i$ are positive definite matrices.

The measurements are given by

$$\underline{y}_i(k) = \underline{C}_i\underline{x}_i(k) + \underline{\theta}_i(k) \quad i=1,\ldots,N \tag{4.3.4}$$

Each controller is allowed only to use his past measurements to find the controls, i.e.,

$$\underline{u}_i(k) = \underline{\gamma}_i^{\,k}(Y_i(k),\ U_i(k-1)) \tag{4.2.5}$$

where

$$Y_i(k) = \{\underline{y}_i(0),\ldots,\underline{y}_i(k)\} \tag{4.2.6}$$

$$U_i(k) = \{\underline{u}_i(0),\ldots,\underline{u}_i(k)\} \tag{4.2.7}$$

It is required to find optimal control strategies $\underline{\gamma}_i^{\,k}$ such that J is minimized.

$\underline{\xi}_i(k)$, k=0,...,T-1 are independent Gaussian variables with zero mean and covariance $\underline{\Xi}_i(k)$.

$\underline{\theta}_i(k)$, k=0,...,T-1 are independent Gaussian variables with zero mean and covariance $\underline{\Theta}_i(k)$.

$\underline{x}_i(0)$ is Gaussian with mean $\overline{\underline{x}}_i(0)$ and covariance $\underline{\Sigma}_i(0)$.

$\underline{\xi}_i(k)$, $\underline{\theta}_j(k)$, $\underline{x}_h(0)$, i, j, h = 1,...,N are all mutually independent.

The matrices $\underline{A}_{ii}$, $\underline{A}_{ij}$, $\underline{B}_i$, $\underline{C}_i$, $\underline{Q}_i$, $\underline{R}_i$ can be time-varying but for simplicity of notation, the dependence on k has been omitted.

The general solution to this problem, assuming no communication of the a posteriori information between the controllers, is not known, although several particular cases have been considered [A1, C3]. We propose to solve this problem using the approach suggested in the previous chapter by defining a new kind of optimality.

Problem 4.1:

$$\underline{x}_i(k+1) = \underline{A}_{ii}\underline{x}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k) + \underline{\xi}_i(k) \tag{4.2.8}$$

$$J = \sum_{i=1}^{N} J_i = \sum_{i=1}^{N} E\{\underline{x}_i'(T)\underline{F}_i\underline{x}_i(T) + \sum_{k=0}^{T-1} \underline{x}_i'(k)\underline{Q}_i\underline{x}_i(k) + \underline{u}_i'(k)\underline{R}_i\underline{u}_i(k)\} \tag{4.2.9}$$

$$E\{\underline{v}_i(k) - \sum_{j\neq 1} A_{ij}\underline{x}_j(k)\} = 0 \tag{4.2.10}$$

$$\underline{u}_i(k) = \underline{\gamma}_i^k(Y_i(k), U_i(k-1); \tilde{I}) \tag{4.2.11}$$

$$\underline{v}_i(k) = \underline{\eta}_i^k(\tilde{I}) \tag{4.2.12}$$

$\tilde{I}$ consists of the a priori information contained in this model. It is required to find $\underline{\gamma}_i^k$ and $\underline{\eta}_i^k$ such that J is minimized.

Using the results of Section 3.5, we obtain the following two-level problem.

Lower Level: (Problem 4.2)

$$\underline{x}_i(k+1) = \underline{A}_{ii}\underline{x}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k) + \underline{\xi}_i(k)$$

$$\underline{u}_i(k) = \underline{\gamma}_i^k(Y_i(k),\ U_i(k-1);\ \tilde{I})$$

$$\underline{v}_i(k) = \underline{\eta}_i^k(\tilde{I})$$

$$\tilde{J}_i = E\{\underline{x}_i'(T)\underline{F}_i\underline{x}_i(T) + \sum_{k=0}^{T-1} \underline{x}_i'(k)\underline{Q}_i\underline{x}_i(k) + \underline{u}'(k)\underline{R}_i\underline{u}_i(k) \tag{4.2.13}$$

$$+ \underline{p}_i'(k)\underline{v}_i(k) - \tilde{\underline{p}}_i'(k)\underline{x}_i(k)\}$$

$$\tilde{\underline{p}}_i(k) = \sum_{j\neq i} \underline{A}_{ji}'\ \underline{p}_j(k) \tag{4.2.14}$$

It is desired to find $\underline{\gamma}_i^k$, $\underline{\eta}_i^k$ to minimize $\tilde{J}_i(\underline{p})$, $i=1,\ldots,N$.

<u>Higher Level: (Problem 4.3)</u>

$$\underset{\underline{p}}{\text{Maximize}} \sum_{i=1}^{N} \tilde{J}_i^*(\underline{p}) \tag{4.2.15}$$

where $\tilde{J}^*(\underline{p})$ is the optimal cost in Problem 4.2 for a particular $\underline{p}$.

3. Structure of the Decomposition

In this section we summarize the relevant equations needed by the lower level controllers and the coordinator.

The optimal control of the ith controller is given by

$$\underline{u}_i^*(k) = -\underline{D}_i(k+1)(\hat{\underline{x}}_i(k) - \overline{\underline{x}}_i(k)) - \underline{E}_i(k+1)\ \underline{p}_i(k) \tag{4.3.1}$$

The gain matrices are given by:

$$\underline{D}_i(k+1) = \underline{T}_i^{-1}(k+1)\ \underline{B}_i'\underline{K}_i(k+1)\ \underline{A}_{ii} \tag{4.3.2}$$

$$\underline{T}_i(k+1) = \underline{R}_i + \underline{B}_i'\underline{K}_i(k+1)\ \underline{B}_i \tag{4.3.3}$$

$$\underline{K}_i(k) = \underline{Q}_i + \underline{A}_{ii}'\underline{K}_i(k+1)\underline{A}_{ii} - \underline{A}_{ii}'\underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1)\underline{A}_{ii}$$

$$\underline{K}_i(T) = \underline{F}_i \tag{4.3.4}$$

$$\underline{E}_i(k+1) = \frac{1}{2}\underline{R}_i^{-1}(k)\ \underline{B}_i'(k) \tag{4.3.5}$$

$$\underline{S}_i(k+1) = \underline{K}_i(k+1) - \underline{K}_i(k+1)\ \underline{B}_i\underline{T}_i^{-1}(k+1)\ \underline{B}_i'\underline{K}_i(k+1) \tag{4.3.6}$$

The estimates $\hat{\underline{x}}_i(k)$ and $\overline{\underline{x}}_i(k)$ are generated as follows.

$$\hat{\underline{x}}_i(k+1) \triangleq E\{\underline{x}_i(k+1)\,|\,Y_i(k+1), U_i(k)\}$$

$$= \underline{A}_{ii}\hat{\underline{x}}_i(k)+\underline{v}_i^*(k)+\underline{B}_i\underline{u}_i^*(k)+\underline{G}_i(k+1)\,[\underline{y}_i(k+1)-\underline{C}_i(\underline{A}_{ii}\hat{\underline{x}}_i(k)+\underline{v}_i^*(k)+\underline{B}_i\underline{u}_i(k))]$$

$$\hat{\underline{x}}_i(0) = \overline{\underline{x}}_i(0) \tag{4.3.7}$$

where

$$\underline{G}_i(k+1) = \underline{\Sigma}_i(k+1|k)\underline{C}_i'\,[\underline{C}_i\underline{\Sigma}_i(k+1|k)\underline{C}_i' + \underline{\Theta}_i(k+1)]^{-1} \tag{4.3.8}$$

$$\underline{\Sigma}_i(k+1|k) = \underline{\Xi}_i(k)+\underline{A}_{ii}[\underline{\Sigma}_i(k|k-1)-\underline{\Sigma}_i(k|k-1)\underline{C}_i'(\underline{C}_i\underline{\Sigma}_i(k|k-1)\underline{C}_i'+\underline{\Theta}_i(k))^{-1}$$

$$\underline{C}_i\Sigma_i(k|k-1)]\underline{A}_{ii}'$$

$$\underline{\Sigma}_i(0|-1) = \underline{\Sigma}_i(0) \tag{4.3.9}$$

$$\overline{\underline{x}}_i(k+1) = E\{\underline{x}_i(k+1)\}$$

$$= -\underline{K}_i^{-1}(k+1)[\underline{r}_i(k+1) + \frac{1}{2}\underline{p}_i(k)] \tag{4.3.10}$$

$\underline{v}_i^*(k)$ and $\underline{r}_i(k)$ are given by

$$\underline{v}_i^*(0) = -\underline{A}_{ii}\overline{\underline{x}}_i(0) - \underline{K}_i^{-1}(1)\underline{r}_i(1) - \frac{1}{2}\underline{S}_i^{-1}(1)\underline{p}_i(0) \tag{4.3.11}$$

$$\underline{v}_i^*(k) = \underline{A}_{ii}\underline{K}_i^{-1}(k)[\underline{r}_i(k) + \frac{1}{2}\underline{p}_i(k-1)] - \underline{K}_i^{-1}(k+1)\underline{r}_i(k+1)$$

$$- \frac{1}{2}\underline{S}_i^{-1}(k+1)\underline{p}_i(k) \qquad k=1,\ldots,T-1 \tag{4.3.12}$$

$$\underline{r}_i(0) = -\frac{1}{2}\tilde{\underline{p}}_i(0) - \frac{1}{2}\underline{A}_{ii}'\underline{p}_i(0) - \underline{A}_{ii}'\underline{S}_i(1)\underline{A}_{ii}\overline{\underline{x}}_i(0) \tag{4.3.13}$$

$$\underline{Q}_i\underline{K}_i^{-1}(k)\underline{r}_i(k) = -\frac{1}{2}\tilde{\underline{p}}_i(k) - \frac{1}{2}\underline{A}_{ii}'\underline{p}_i(k) + \frac{1}{2}\underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii}\underline{K}_i^{-1}(k)\underline{p}_i(k-1)$$

$$k=1,\ldots,T-1 \tag{4.3.14}$$

$$\underline{r}_i(T) = \underline{0} \tag{4.3.15}$$

The structure of the control mechanism is illustrated in Fig. 4.1.

The gain matrices can all be computed off-line, along with $\underline{r}_i(k)$ and $\underline{v}_i^*(k)$, which depend on $\underline{p}(k)$. $\underline{K}_i(k)$ is the solution of the Riccati equation assuming the systems are uncoupled and $\underline{D}_i(k)$ is the optimal gain matrix for each of the uncoupled deterministic optimal problems.

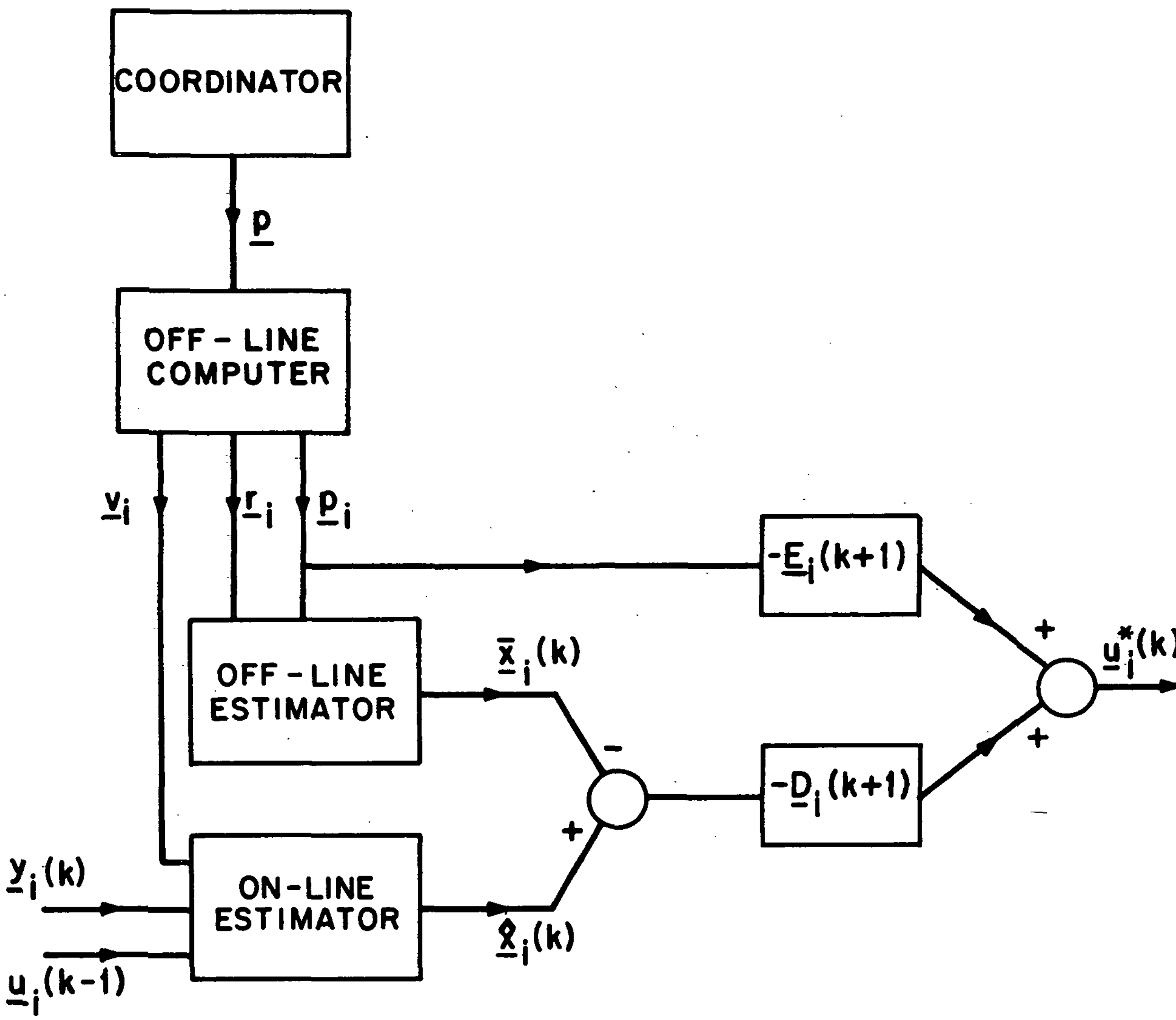

Fig. 4.1 Structure of Control for i^{TH} Controller

$\bar{\underline{x}}_i(k)$ is the unconditional mean of $\underline{x}_i(k)$ by the ith controller given only his a priori information. It can be computed off-line given $\underline{r}_i(k)$ and $\underline{p}_i(k)$.

$\hat{\underline{x}}_i(k)$ is the best estimate of $\underline{x}_i(k)$ given the measurements of the ith controller and his a priori information. It is generated using $\underline{v}_i{}^*(k)$ calculated off-line and the on-line measurements $\underline{u}_i{}^*(k)$ and $\underline{y}_i(k)$.

The coordinator finds the optimal $\underline{p}^*(k)$'s by solving the following deterministic two-point boundary value problem

$$\bar{\underline{x}}(k+1) = \underline{A}\,\bar{\underline{x}}(k) - \frac{1}{2}\underline{B}\,\underline{R}^{-1}\underline{B}'\underline{\lambda}(k+1) \tag{4.3.16}$$

$$\underline{\lambda}(k) = \underline{A}'\underline{\lambda}(k+1) + 2\underline{Q}\,\bar{\underline{x}}(k) \tag{4.3.17}$$

$$\bar{\underline{x}}(0) \text{ given}$$

$$\underline{\lambda}(T) = 2\underline{F}\,\underline{x}(T) \tag{4.3.18}$$

$$\underline{p}^*(k) = -\underline{\lambda}(k+1) \qquad k=0,\dots,T-1 \tag{4.3.19}$$

The matrices $\underline{A}$, $\underline{B}$ and $\underline{Q}$ are as defined in Section 5. $\underline{R}$ and $\underline{F}$ are given by

$$\underline{R} \triangleq \begin{bmatrix} \underline{R}_1 & \underline{0} & \cdots & \\ \underline{0} & \underline{R}_2 & \cdots & \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \underline{R}_N \end{bmatrix} \qquad \underline{F} \triangleq \begin{bmatrix} \underline{F}_1 & \underline{0} & 0 & \cdots & \\ \underline{0} & \underline{F}_2 & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \underline{F}_N \end{bmatrix}$$

Alternatively, $\underline{p}^*(k)$ can be expressed as follows.

$$\underline{p}^*(k) = -2\underline{K}(k+1)\bar{\underline{x}}(k+1) \tag{4.3.20}$$

where

$$\bar{\underline{x}}(k+1) = (\underline{A}-\underline{B}\,\underline{T}^{-1}(k+1)\underline{B}'\underline{K}(k+1)\underline{A})\bar{\underline{x}}(k)$$

$$\bar{\underline{x}}(0) \text{ given} \tag{4.3.21}$$

$$\underline{K}(k) = \underline{Q} + \underline{A}'\underline{K}(k+1)[\underline{I}-\underline{B}\,\underline{T}^{-1}(k+1)\underline{B}'\underline{K}(k+1)]\underline{A}$$

$$\underline{K}(T) = \underline{F} \tag{4.3.22}$$

$$\underline{T}(k+1) = \underline{R} + \underline{B}'\underline{K}(k+1)\underline{B} \tag{4.3.23}$$

4. Solution of the Lower Level Problem

Since each controller knows the structure of his system as defined in Problem 4.2 we shall not include the a priori information in specifying the information structure of the controller. Thus $\underline{u}_i(k)$ would depend on $Y_i(k)$ and $U_i(k-1)$ while $\underline{v}_i(k)$ is allowed to depend on the a priori information only.

The problem as stated has a nonquadratic cost functional and controls which depend on different information sets. However, the information of $\underline{v}_i(k)$ consists of a priori information only and thus is included in that of $\underline{u}_i(k)$. This makes things easier than the general dynamic team problem and the following theorem can be used.

Theorem 4.4.1:

Consider the system

$$\underline{x}(k+1) = \underline{f}(\underline{x}(k), \underline{u}(k), \underline{v}(k), \underline{\xi}(k)) \tag{4.4.1}$$

$$\underline{u}(k) = \underline{\gamma}^k(Y(k)) \tag{4.4.2}$$

$$\underline{v}(k) = \underline{\eta}^k(Z(k)) \tag{4.4.3}$$

$$Z(k) \subset Y(k) \tag{4.4.4}$$

$\underline{\xi}(k)$ is a white noise process driving the system and $Y(k)$, $Z(k)$ are information available to the controller.

$$Y(k) = \{\underline{y}(0),\ldots,\underline{y}(k);\ \underline{u}(0),\ldots,\underline{u}(k-1);\ \underline{v}(0),\ldots,\underline{v}(k-1)\} \tag{4.4.5}$$

$$\underline{y}(k) = \underline{h}(\underline{x}(k), \underline{\theta}(k)) \tag{4.4.6}$$

$\underline{\theta}(k)$ is a white noise process.

$$J = E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k),\ \underline{u}(k),\ \underline{v}(k))\} \tag{4.4.7}$$

Then the optimal cost is given by

$$E\{V(Y(0),0)\} \tag{4.4.8}$$

where $V(Y(k),k))$ satisfies the functional equation

$$V(Y(k),k) = \min_{\substack{\underline{u}(k) \\ \underline{\eta}^k}} E\{L(\underline{x}(k),\ \underline{u}(k),\ \underline{v}(k)) + V(Y(k+1),k+1)\,|\,Y(k)\} \tag{4.4.9}$$

$$V(Y(T),T) = E\{K(x(T))\,|\,Y(T)\} \tag{4.4.10}$$

__Proof:__

Define

$$V(Y(k),k) = \min_{\substack{\underline{u}(k),\underline{\gamma}^{k+1},\ldots,\underline{\gamma}^{T-1} \\ \underline{\eta}^{k},\underline{\eta}^{k+1},\ldots,\underline{\eta}^{T-1}}} E\{\sum_{t=k}^{T-1} L(\underline{x}(t),\underline{u}(t),\underline{v}(t)) + K(\underline{x}(T))\,|\,Y(k)\}$$

$$= \min_{\substack{\underline{u}(k) \\ \underline{\eta}^k}} \{E\{L(\underline{x}(k),\underline{u}(k),\underline{v}(k))\,|\,Y(k)\}$$

$$+ \min_{\substack{\underline{\gamma}^{k+1},\ldots,\underline{\gamma}^{T-1} \\ \underline{\eta}^{k+1},\ldots,\underline{\eta}^{T-1}}} E\{\sum_{t=k+1}^{T-1} L(\underline{x}(t),\underline{u}(t),\underline{v}(t)) + K(\underline{x}(T))\,|\,Y(k)\}\} \tag{4.4.11}$$

Note that the minimization is done with respect to $\underline{u}(k)$, and the control strategies $\underline{\gamma}^{k+1},\ldots,\underline{\gamma}^{T-1}$, $\underline{\eta}^{k},\ldots,\underline{\eta}^{T-1}$. The first term in the minimization is separated from the rest because it does not depend on $\underline{\gamma}^{k+1},\ldots,\underline{\gamma}^{T-1}$, $\underline{\eta}^{k+1},\ldots,\underline{\eta}^{T-1}$.

Using Lemma A.3 (in Appendix A),

$$\min_{\substack{\underline{\Upsilon}^{k+1},\dots,\underline{\Upsilon}^{T-1}\\ \underline{\eta}^{k+1},\dots,\underline{\eta}^{T-1}}} E\{\sum_{t=k+1}^{T-1} L(\underline{x}(t),\underline{u}(t),\underline{v}(t)) + K(\underline{x}(T))\,|\,Y(k)\}$$

$$= E\{\min_{\substack{\underline{u}(k+1),\underline{\Upsilon}^{k+2},\dots,\underline{\Upsilon}^{T-1}\\ \underline{\eta}^{k+1},\dots,\underline{\eta}^{T-1}}} E\{\sum_{t=k+1}^{T-1} L(\underline{x}(t),\underline{u}(t),\underline{v}(t)) + K(x(T))\,|\,Y(k+1)\}\,|\,Y(k)\}$$

$$= E\{V(Y(k+1),k+1)\,|\,Y(k)\} \qquad (4.4.12)$$

From this and equation (4.4.11) we obtain equation (4.4.9) and further

$$V(Y(0),0) = \min_{\substack{\underline{u}(0),\underline{\Upsilon}^{1},\dots,\underline{\Upsilon}^{T-1}\\ \underline{\eta}^{0},\underline{\eta}^{1},\dots,\underline{\eta}^{T-1}}} E\{\sum_{k=0}^{T-1} L(\underline{x}(k),\underline{u}(k),\underline{v}(k)) + K(\underline{x}(T))\,|\,Y(0)\}$$

Again by Lemma A.3

$$\min_{\substack{\underline{\Upsilon}^{0},\dots,\underline{\Upsilon}^{T-1}\\ \underline{\eta}^{0},\dots,\underline{\eta}^{T-1}}} E\{\sum_{k=0}^{T-1} L(\underline{x}(k),\underline{u}(k),\underline{v}(k)) + K(\underline{x}(T))\}$$

$$= E\{\min_{\substack{\underline{u}(0),\underline{\Upsilon}^{1},\dots,\underline{\Upsilon}^{T-1}\\ \underline{\eta}^{0},\underline{\eta}^{1},\dots,\underline{\eta}^{T-1}}} E\{\sum_{k=0}^{T-1} L(\underline{x}(k),\underline{u}(k),\underline{v}(k)) + K(\underline{x}(T))\,|\,Y(0)\}\}$$

$$= E\{V(T(0),0)\}. \qquad (4.4.14)$$

We can then apply this theorem to solve the lower-level problem. This will be stated in the following theorem. Since we have a linear system, with Gaussian driving and observation noises, the information Y(k) can be

replaced by the sufficient statistics $\hat{\underline{x}}_i(k)$. From now on we would deal with $V_i(\hat{\underline{x}}_i(k),k)$ instead of $V_i(Y_i(k),U_i(k-1),k)$.

Theorem 4.4.2:

The solution to the lower level problem is given by

$$\underline{u}_i^*(k) = -\underline{D}_i(k+1)(\hat{\underline{x}}_i(k) - \overline{\underline{x}}_i(k)) - \underline{E}_i(k+1)\ \underline{p}_i(k) \tag{4.4.15}$$

$$\underline{v}_i^*(0) = -\underline{A}_{ii}\overline{\underline{x}}_i(0) - \underline{K}_i^{-1}(1)\underline{r}_i(1) - \frac{1}{2}\underline{S}_i^{-1}(1)\underline{p}_i(0) \tag{4.4.16}$$

$$\underline{v}_i^*(k) = \underline{A}_{ii}\underline{K}_i^{-1}(k)[\underline{r}_i(k) + \frac{1}{2}\underline{p}_i(k-1)] - \underline{K}_i^{-1}(k+1)\underline{r}_i(k+1)$$

$$- \frac{1}{2}\underline{S}_i^{-1}(k+1)\underline{p}_i(k) \qquad k=1,\ldots,T-1 \tag{4.4.17}$$

where $\underline{D}_i(k)$, $\underline{E}_i(k)$, $\hat{\underline{x}}_i(k)$, $\overline{\underline{x}}_i(k)$, $\underline{r}_i(k)$, $\underline{K}_i(k)$ and $\underline{S}_i(k)$ are as given in Section 3. Moreover, the optimal cost is given by $E\{V_i(\hat{\underline{x}}_i(0),0)\}$ where

$$V_i(\hat{\underline{x}}_i(k),k) = \hat{\underline{x}}_i'(k)\underline{K}_i(k)\hat{\underline{x}}_i(k) + 2\underline{r}_i'(k)\hat{\underline{x}}_i(k) + s_i(k) \tag{4.4.18}$$

with

$$s_i(k) = s_i(k+1) + 2\underline{r}_i'(k+1)\underline{v}_i^*(k) + \underline{v}_i^{*\prime}(k)\underline{K}_i(k+1)\underline{v}_i^*(k)$$

$$- [\underline{K}_i(k+1)\underline{v}_i^*(k) + \underline{r}_i(k+1)]'\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i'[\underline{K}_i(k+1)\underline{v}_i^*(k) + \underline{r}_i(k+1)]$$

$$+ \underline{p}_i'(k)\underline{v}_i^*(k) + \mathrm{tr}\underline{Q}_i\underline{\Sigma}_i(k|k) + \mathrm{tr}\underline{K}_i(k+1)(\underline{\Sigma}_i(k+1|k)-\underline{\Sigma}_i(k+1|k+1))$$

$$s_i(T) = \mathrm{tr}\underline{F}_i\underline{\Sigma}_i(T|T) \tag{4.4.19}$$

$$\underline{\Sigma}_i(k|k) = \underline{\Sigma}_i(k|k-1) - \underline{\Sigma}_i(k|k-1)\underline{C}_i'[\underline{C}_i\underline{\Sigma}_i(k|k-1)\underline{C}_i' + \underline{\Theta}_i]^{-1}\underline{C}_i\underline{\Sigma}_i(k|k-1) \tag{4.4.20}$$

$$\underline{\Sigma}_i(k+1|k) = \underline{A}_{ii}\underline{\Sigma}_i(k|k)\underline{A}_{ii}' + \underline{\Xi}(k)$$

$$\underline{\Sigma}_i(0|-1) = \underline{\Sigma}_i(0) \tag{4.4.21}$$

<u>Proof</u>:

The functional equation corresponding to this problem is

$$V_i(\hat{\underline{x}}_i(k),k) = \underset{\substack{\underline{u}_i(k)\\ \underline{v}_i(k)}}{\text{Min}}\ E\{\underline{x}_i'(k)\underline{Q}_i\underline{x}_i(k) - \tilde{\underline{p}}_i'(k)\underline{x}_i(k) + \underline{u}_i'(k)\underline{R}_i\underline{u}_i(k)$$

$$+ \underline{p}_i'(k)\underline{v}_i(k) + V_i(\hat{\underline{x}}_i(k+1),k+1)\,|\,\hat{\underline{x}}_i(k)\} \tag{4.4.22}$$

where $\underline{v}_i(k)$ is to be independent of any a posteriori information.

If we let $V_i(\hat{\underline{x}}_i(k),k)$ to be of the form given by equation (4.4.18), the right-hand side of (4.4.22) becomes

$$E\{\underline{x}_i'(k)\underline{Q}_i\underline{x}_i(k) - \tilde{\underline{p}}_i'(k)\underline{x}_i(k) + \underline{u}_i'(k)\underline{R}_i\underline{u}_i(k) + \underline{p}_i'(k)\underline{v}_i(k)$$

$$+ \hat{\underline{x}}_i'(k+1)\underline{K}_i(k+1)\hat{\underline{x}}_i(k+1) + 2\underline{r}_i'(k+1)\hat{\underline{x}}_i(k+1) + s_i(k+1)\,|\,\hat{\underline{x}}_i(k)\}$$

$$= \hat{\underline{x}}_i'(k)\underline{Q}_i\hat{\underline{x}}_i(k) + \text{tr}\underline{Q}_i\underline{\Sigma}_i(k|k) - \tilde{\underline{p}}_i'(k)\underline{x}_i(k) + \underline{u}_i'(k)\underline{R}_i\underline{u}_i(k) + \underline{p}_i'(k)\underline{v}_i(k)$$

$$+ [\underline{A}_{ii}\hat{\underline{x}}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k)]'\underline{K}_i(k+1)[\underline{A}_{ii}\hat{\underline{x}}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k)]$$

$$+ \text{tr}\underline{K}_i(k+1)\underline{\Sigma}_i(k+1|k) + 2\underline{r}_i'(k+1)[\underline{A}_{ii}\hat{\underline{x}}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k)] + s_i(k+1)$$

$$- \text{tr}\underline{K}_i(k+1)\underline{\Sigma}_i(k+1|k+1) \tag{4.4.23}$$

where we have used the fact that

$$E\{\hat{\underline{x}}_i'(k+1)\underline{K}_i(k+1)\hat{\underline{x}}_i(k+1)\,|\,\hat{\underline{x}}_i(k)\} = [\underline{A}_{ii}\hat{\underline{x}}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k)]'\underline{K}_i(k+1)$$

$$[\underline{A}_{ii}\hat{\underline{x}}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k)] + \text{tr}\underline{K}_i(k+1)(\underline{\Sigma}_i(k+1|k) - \underline{\Sigma}_i(k+1|k+1)) \tag{4.4.24}$$

Given $\underline{v}_i(k)$, minimizing (4.4.23) with respect to $\underline{u}_i(k)$ gives

$$\underline{u}_i^*(k) = -\underline{T}_i^{-1}(k+1)\underline{B}_i'[\underline{K}_i(k+1)\underline{A}_{ii}\hat{\underline{x}}_i(k) + \underline{K}_i(k+1)\underline{v}_i(k) + \underline{r}_i(k+1)] \tag{4.4.25}$$

Denote (4.4.23) with $\underline{u}_i^*$ substituted in by $\underline{W}_i(\hat{\underline{x}}_i(k),k)$. To minimize with respect to $\underline{v}_i(k)$ we minimize $E\{\underline{W}_i(\hat{\underline{x}}_i(k),k\}$. This gives

$$\underline{p}_i(k) + 2\underline{K}_i(k+1)[\underline{A}_{ii}\overline{\underline{x}}_i(k) + \underline{B}_i\overline{\underline{u}}_i^*(k)] + 2\underline{K}_i(k+1)\underline{v}_i^*(k) + 2\underline{r}_i(k+1) = 0 \tag{4.4.26}$$

where

$$\begin{aligned}\overline{\underline{u}}_i^*(k) &= E\{\underline{u}_i^*(k)\} \\ &= -\underline{T}_i^{-1}(k+1)\underline{B}_i'[\underline{K}_i(k+1)\underline{A}_{ii}\overline{\underline{x}}_i(k) + \underline{K}_i(k+1)\underline{v}_i(k) + \underline{r}_i(k+1)]\end{aligned} \tag{4.4.27}$$

Substituting equation 4.4.27 into equation 4.4.26 we have

$$\begin{aligned}&[\underline{I} - \underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i']\underline{K}_i(k+1)\underline{v}_i^*(k) \\ &= -[\underline{I} - \underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i'][\underline{K}_i(k+1)\underline{A}_{ii}\overline{\underline{x}}_i(k) + \underline{r}_i(k+1)] - \frac{1}{2}\underline{p}_i(k)\end{aligned} \tag{4.4.28}$$

Since $\underline{K}_i(k+1)$ is invertible (see Appendix B)

$$\begin{aligned}\underline{S}_i(k+1) &= \underline{K}_i(k+1) - \underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1) \\ &= [\underline{K}_i^{-1}(k+1) + \underline{B}_i\underline{R}_i^{-1}\underline{B}_i']^{-1}\end{aligned} \tag{4.4.29}$$

$[\underline{I}-\underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i'] = \underline{S}_i(k+1)\underline{K}_i^{-1}(k+1)$ is then invertible. Thus

$$\underline{v}_i^*(k) = -\underline{A}_{ii}\overline{\underline{x}}_i(k) - \underline{K}_i^{-1}(k+1)\underline{r}_i(k+1) - \frac{1}{2}\underline{S}_i^{-1}(k+1)\underline{p}_i(k) \tag{4.4.30}$$

$$\begin{aligned}\underline{u}_i^*(k) = &-\underline{T}_i^{-1}(k+1)\underline{B}_i'[\underline{K}_i(k+1)\underline{A}_{ii}(\hat{\underline{x}}_i(k) - \overline{\underline{x}}_i(k)) \\ &- \frac{1}{2}(\underline{I} - \underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i')^{-1}\underline{p}_i(k)]\end{aligned} \tag{4.4.31}$$

It can be shown that (see Appendix B)

$$\underline{T}_i^{-1}(k+1)\underline{B}_i'(\underline{I} - \underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i')^{-1} = \underline{R}_i^{-1}\underline{B}_i' \tag{4.4.32}$$

Thus

$$\underline{u}_i^*(k) = -\underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1)\underline{A}_{ii}(\hat{\underline{x}}_i(k) - \overline{\underline{x}}_i(k)) + \frac{1}{2}\underline{R}_i^{-1}\underline{B}_i'\underline{p}_i(k) \tag{4.4.33}$$

By substitution into equation 4.4.22 and identifying the terms quadratic in $\hat{\underline{x}}_i(k)$, linear in $\hat{\underline{x}}_i(k)$ and independent of $\hat{\underline{x}}_i(k)$ we obtain equations for $\underline{K}_i(k)$, $S_i(k)$ as well as

$$\underline{r}_i(k) = -\frac{1}{2}\tilde{\underline{p}}_i(k) + \underline{A}_{ii}[\underline{I}_i - \underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i']\underline{r}_i(k+1)$$
$$+ \underline{A}_{ii}'\underline{S}_i(k+1)\underline{v}_i^*(k)$$
$$\underline{r}_i(T) = \underline{0} \qquad (4.4.34)$$

To find the optimal controls $\underline{u}_i^*(k)$ and the optimal "estimates" $\underline{v}_i^*(k)$, a two-point boundary value problem has to be solved. This involves equations 4.4.30, 4.4.34, and the following equation

$$\overline{\underline{x}}_i(k+1) = \underline{A}_{ii}\overline{\underline{x}}_i(k) + \underline{v}_i^*(k) + \underline{B}_i\overline{\underline{u}}_i^*(k) \qquad (4.4.35)$$
$$\overline{\underline{x}}_i(0) \text{ given}$$

From equation 4.4.33

$$\overline{\underline{u}}_i^*(k) = \frac{1}{2}\underline{R}_i^{-1}\underline{B}_i'\underline{p}_i(k) \qquad (4.4.36)$$

Substitution of (4.4.30) and (4.4.36) into (4.4.35) yields

$$\overline{\underline{x}}_i(k+1) = -\underline{K}_i^{-1}(k+1)[\underline{r}_i(k+1) + \frac{1}{2}\underline{p}_i(k)] \qquad (4.4.37)$$

From these we obtain equations 4.4.16 and 4.4.17. Substitution of (4.4.16) into (4.4.34) yields (4.3.13). Substitution of (4.4.17) into (4.4.34) gives

$$[\underline{I}_i - \underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii}\underline{K}_i^{-1}(k)]\underline{r}_i(k) = -\frac{1}{2}\tilde{\underline{p}}_i(k) - \frac{1}{2}\underline{A}_{ii}'\underline{p}_i(k)$$
$$+ \frac{1}{2}\underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii}\underline{K}_i^{-1}(k)\ \underline{p}_i(k-1) \qquad (4.4.38)$$

Since from the Riccati equation

$$\underline{I}_i - \underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii}\underline{K}_i^{-1}(k) = \underline{Q}_i\underline{K}_i^{-1}(k) \qquad (4.4.39)$$

we obtain equation 4.3.14. Essentially, the two-point boundary value problem is uncoupled and becomes a single equation in $\underline{r}_i(k)$. $\underline{r}_i(k)$ is uniquely defined when $\underline{Q}_i$ is positive definite. This is a sufficient condition for the positive definiteness of $\underline{K}_i(k)$, $k=0,\ldots,T-1$. Q.E.D.

The control $\underline{u}_i^*(k)$ which is actually applied by the ith controller consists of two parts: a closed loop part which depends on the measurements and an open loop part which does not. The closed loop part can be written to depend on the difference between the a priori and the a posteriori estimates of the ith controller about the state of the ith subsystem. It looks like the solution of a tracking problem with $\bar{\underline{x}}_i(k)$ as the reference state. In fact, the optimal cost to go $V_i(\hat{\underline{x}}_i(k),k)$ has a form similar to that of the tracking problem. The open loop part depends only on $\underline{p}$, the coordinating signals received from the higher level. When the a priori and a posteriori estimates of the local controllers are the same, as in the case of no measurements, the closed loop part disappears and only the open loop control remains. In the next section we will find out what the open loop part really is.

5. Solution of the Higher Level Problem

The higher level problem is choosing the optimal $\underline{p}^*$ to maximize

$$\tilde{J}^*(\underline{p}) = \sum_{i=1}^{N} \tilde{J}_i^*(\underline{p})$$

From Section 4,

$$\tilde{J}_i^*(\underline{p}) = \overline{\underline{x}}_i'(0)\underline{K}_i(0)\overline{\underline{x}}_i(0) + 2\underline{r}_i'(0)\overline{\underline{x}}_i(0) + s_i(0) \qquad (4.5.1)$$

where $\underline{r}_i(0)$ satisfies equation 4.3.13 and $s_i(0)$ is given by equation 4.4.19.

Since $\overline{\underline{x}}_i'(0)\underline{K}_i(0)\overline{\underline{x}}_i(0)$ is independent of $\underline{p}$, the higher level problem is

$$\operatorname*{Max}_{\underline{p}} \sum_{i=1}^{N} 2\underline{r}_i'(0)\overline{\underline{x}}_i(0) + s_i(0) \qquad (4.5.2)$$

Let

$$\underline{r}(k) \triangleq \begin{bmatrix} \underline{r}_1(k) \\ \cdot \\ \cdot \\ \cdot \\ \underline{r}_N(k) \end{bmatrix} \qquad \underline{Q} \triangleq \begin{bmatrix} \underline{Q}_1 & \underline{0} & \cdots & \\ \underline{0} & \underline{Q}_2 & \cdots & \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \underline{Q}_N \end{bmatrix}$$

$$\tilde{\underline{K}}(k) \triangleq \begin{bmatrix} \underline{K}_1(k) & \underline{0} & \cdots & \\ \underline{0} & \underline{K}_2(k) & \cdots & \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \underline{K}_N(k) \end{bmatrix} \qquad \underline{A} \triangleq \begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} & \cdots & \\ \underline{A}_{21} & \underline{A}_{22} & \cdots & \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \underline{A}_{NN} \end{bmatrix}$$

$$\underline{B} \triangleq \begin{bmatrix} \underline{B}_1 & \underline{0} & \cdots & \\ \underline{0} & \underline{B}_2 & \cdots & \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \underline{B}_N \end{bmatrix} \qquad \tilde{\underline{S}}(k) \triangleq \begin{bmatrix} \underline{S}_1(k) & \underline{0} & \cdots & \\ \underline{0} & \underline{S}_2(k) & \cdots & \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \underline{S}_N(k) \end{bmatrix}$$

Then equations 4.3.13 and 4.3.14 become

$$\underline{r}(0) = -\frac{1}{2}\underline{A}'\underline{p}(0) - (\underline{I} - \underline{Q}\,\tilde{\underline{K}}^{-1}(0))\tilde{\underline{K}}(0)\overline{\underline{x}}(0) \qquad (4.5.4)$$

$$\underline{Q}\,\underline{\tilde{K}}^{-1}(k)\underline{r}(k) = -\frac{1}{2}\underline{A}'\underline{p}(k) + \frac{1}{2}(\underline{I} - \underline{Q}\,\underline{\tilde{K}}^{-1}(k))\underline{p}(k-1) \qquad (4.5.5)$$

$$k=1,\ldots,T-1$$

$$s_i(0) = \sum_{k=0}^{T-1}(s_i(k) - s_i(k+1)) + s_i(T)$$

$$= \underline{\bar{x}}_i'(0)\underline{A}_{ii}'\underline{S}_i(1)\underline{A}_{ii}\underline{\bar{x}}_i(0) + \mathrm{tr}\ \underline{K}_i(T)\underline{\Sigma}_i(T|T)$$

$$+ \sum_{k=0}^{T-1}\{\mathrm{tr}\ \underline{Q}_i\underline{\Sigma}_i(k|k) + \mathrm{tr}\ \underline{K}_i(k+1)(\underline{\Sigma}_i(k+1|k) - \underline{\Sigma}_i(k+1|k+1))\}$$

$$+ \sum_{k=1}^{T-1}\{\underline{r}_i'(k)[\underline{K}_i^{-1}(k)\underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii}\underline{K}_i^{-1}(k) - \underline{K}_i^{-1}(k)]\underline{r}_i(k)$$

$$+ \underline{p}_i'(k-1)[\underline{K}_i^{-1}(k)\underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii}\underline{K}_i^{-1}(k) - \underline{K}_i^{-1}(k)]\underline{r}_i(k)$$

$$+ \frac{1}{4}\underline{p}_i'(k-1)[\underline{K}_i^{-1}(k)\underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii}\underline{K}_i^{-1}(k) - \underline{S}_i^{-1}(k)]\underline{p}_i(k-1)\}$$

$$- \underline{r}_i'(T)\underline{K}_i^{-1}(T)\underline{r}_i(T) - \underline{p}_i'(T-1)\underline{K}_i^{-1}(T)\underline{r}_i(T)$$

$$- \frac{1}{4}\underline{p}_i'(T-1)\underline{S}_i^{-1}(T)\underline{p}_i(T-1) \qquad (4.5.6)$$

The terms involving $\underline{\Sigma}_i(k|k)$ and $\underline{\Sigma}_i(k+1|k)$ are independent of $\underline{p}$. Thus the quantity to be maximized is

$$2\underline{r}(0)\underline{\bar{x}}(0) + \sum_{k=1}^{T-1} - \underline{r}'(k)\underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k)\underline{r}(k) - \underline{p}'(k-1)\underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k)\underline{r}(k)$$

$$- \frac{1}{4}\underline{p}'(k-1)[\underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k) + \underline{B}\,\underline{R}^{-1}\underline{B}']\,\underline{p}(k-1) - \frac{1}{4}\underline{p}(T-1)\underline{\tilde{S}}^{-1}(T)\underline{p}(T-1)$$

$$(4.5.7)$$

Redefining

$$\underline{\lambda}(k) = -\underline{p}(k-1) \qquad k=1,\ldots,T$$

we have

$$\text{Max } \underline{\lambda}'(1)\underline{A}\ \overline{\underline{x}}(0) + \sum_{k=1}^{T-1} - \underline{r}'(k)\underline{\tilde{K}}^{-1}(k)\underline{Q}\ \underline{\tilde{K}}^{-1}(k)\underline{r}(k) + \underline{\lambda}'(k)\underline{\tilde{K}}^{-1}(k)\underline{Q}\ \underline{\tilde{K}}^{-1}(k)\underline{r}(k)$$

$$- \frac{1}{4}\underline{\lambda}'(k)\ [\underline{\tilde{K}}^{-1}(k)\underline{Q}\ \underline{\tilde{K}}^{-1}(k) + \underline{B}\ \underline{R}^{-1}\underline{B}']\underline{\lambda}(k) - \frac{1}{4}\underline{\lambda}'(T)\underline{\tilde{S}}^{-1}(T)\underline{\lambda}(T) \quad (4.5.8)$$

with respect to

$$\underline{r}(k); \quad k=1,\ldots,T-1$$

$$\underline{\lambda}(k); \quad k=1,\ldots,T$$

such that

$$\underline{Q}\ \underline{\tilde{K}}^{-1}(k)\underline{r}(k) = \frac{1}{2}\underline{A}'\underline{\lambda}(k+1) - \frac{1}{2}[\underline{I} - \underline{Q}\ \underline{\tilde{K}}^{-1}(k)]\ \underline{\lambda}(k)$$

$$k=1,\ldots,T-1 \qquad (4.5.9)$$

<u>Theorem 4.5.1</u>:

The optimal solution $\underline{\lambda}^*(k)$ to equations 4.5.8 and 4.5.9 corresponds to the costates of the deterministic linear regulator problem for the entire system. Minimize

$$\underline{x}'(T)\underline{F}\ \underline{x}(T) + \sum_{k=0}^{T-1} \underline{x}'(k)\ \underline{Q}\ \underline{x}(k) + \underline{u}'(k)\ \underline{R}\ \underline{u}(k) \qquad (4.5.10)$$

subject to

$$\underline{x}(k+1) = \underline{A}\ \underline{x}(k) + \underline{B}\ \underline{u}(k)$$

$$\underline{x}(0) = \overline{\underline{x}}(0) \text{ given} \qquad (4.5.11)$$

<u>Proof</u>:

We form the Lagrangian $H(\underline{\lambda},\underline{r},\underline{\alpha})$ given by

$$H(\underline{\lambda},\underline{r},\underline{\alpha}) = \underline{\lambda}'(1)\underline{A}\ \overline{\underline{x}}(0) + \sum_{k=0}^{T-1} -\underline{r}'(k)\underline{\tilde{K}}^{-1}(k)\underline{Q}\ \underline{\tilde{K}}^{-1}(k)\underline{r}(k)$$

$$+ \underline{\lambda}'(k)\underline{\tilde{K}}^{-1}(k)\underline{Q}\ \underline{\tilde{K}}^{-1}(k)\underline{r}(k) - \frac{1}{4}\underline{\lambda}'(k)\ [\underline{\tilde{K}}^{-1}(k)\underline{Q}\ \underline{\tilde{K}}^{-1}(k)+\underline{B}\ \underline{R}^{-1}\underline{B}']\ \underline{\lambda}(k)$$

$$- \underline{\alpha}'(k)\ [\underline{Q}\ \underline{\tilde{K}}^{-1}(k)\underline{r}(k) - \frac{1}{2}\underline{A}'\underline{\lambda}(k+1) + \frac{1}{2}(\underline{I} - \underline{Q}\ \underline{\tilde{K}}^{-1}(k))\underline{\lambda}(k)]$$

$$- \frac{1}{4}\underline{\lambda}'(T)\underline{\tilde{S}}^{-1}(T)\underline{\lambda}(T) \qquad (4.5.12)$$

Using the necessary conditions for optimality we obtain

$$\frac{\partial H}{\partial \underline{\lambda}(1)} = \underline{A}\,\underline{\bar{x}}(0) + \underline{\tilde{K}}^{-1}(1)\underline{Q}\,\underline{\tilde{K}}^{-1}(1)\underline{r}(1) - \frac{1}{2}[\underline{\tilde{K}}^{-1}(1)\underline{Q}\,\underline{\tilde{K}}^{-1}(1) + \underline{B}\,\underline{R}^{-1}\underline{B}']\,\underline{\lambda}(1)$$
$$- \frac{1}{2}[\underline{I} - \underline{Q}\,\underline{\tilde{K}}^{-1}(1)]'\,\underline{\alpha}(1) = \underline{0} \qquad (4.5.13)$$

$$\frac{\partial H}{\partial \underline{\lambda}(k)} = \underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k)\underline{r}(k) - \frac{1}{2}[\underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k) + \underline{B}\,\underline{R}^{-1}\underline{B}']\underline{\lambda}(k) + \frac{1}{2}\underline{A}\,\underline{\alpha}(k-1)$$
$$- \frac{1}{2}[\underline{I} - \underline{Q}\,\underline{\tilde{K}}^{-1}(k)]'\underline{\alpha}(k) = \underline{0} \qquad k=2,\ldots,T-1 \qquad (4.5.14)$$

$$\frac{\partial H}{\partial \underline{\lambda}(T)} = \frac{1}{2}\underline{A}\,\underline{\alpha}(T-1) - \frac{1}{2}\underline{\tilde{S}}^{-1}(T)\,\underline{\lambda}(T) = \underline{0} \qquad (4.5.15)$$

$$\frac{\partial H}{\partial \underline{r}(k)} = -2\underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k)\underline{r}(k) + \underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k)\underline{\lambda}(k) - \underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\alpha}(k) = \underline{0}$$
$$k=1,\ldots,T-1 \qquad (4.5.16)$$

$$\frac{\partial H}{\partial \underline{\alpha}(k)} = -\underline{Q}\,\underline{\tilde{K}}^{-1}(k)\underline{r}(k) + \frac{1}{2}\underline{A}'\,\underline{\lambda}(k+1) - \frac{1}{2}[\underline{I} - \underline{Q}\,\underline{\tilde{K}}^{-1}(k)]\,\underline{\lambda}(k) = \underline{0}$$
$$k=1,\ldots,T-1 \qquad (4.6.17)$$

From (4.5.13) and (4.5.16), we obtain

$$\underline{\alpha}(1) = 2\,\underline{A}\,\underline{\bar{x}}(0) - \underline{B}\,\underline{R}^{-1}\underline{B}'\underline{\lambda}(1) \qquad (4.5.18)$$

From (4.5.14) and (4.5.16), we obtain

$$\underline{\alpha}(k) = \underline{A}\,\underline{\alpha}(k-1) - \frac{1}{2}\underline{B}\,\underline{R}^{-1}\underline{B}'\underline{\lambda}(k) \quad k=2,\ldots,T-1 \qquad (4.5.19)$$

Since

$$\underline{\tilde{S}}^{-1}(T) = \underline{F}^{-1} + \underline{B}\,\underline{R}^{-1}\underline{B}' \qquad (4.5.20)$$

equation 4.5.15 becomes

$$(\underline{F}^{-1} + \underline{B}\,\underline{R}^{-1}\underline{B}')\underline{\lambda}(T) = \underline{A}\,\underline{\alpha}(T-1)$$
$$\underline{\lambda}(T) = \underline{F}[\underline{A}\,\underline{\alpha}(T-1) - \underline{B}\,\underline{R}^{-1}\underline{B}'\underline{\lambda}(T)] \qquad (4.5.21)$$

From (4.5.17) and (4.5.16), we obtain

$$\underline{\lambda}(k) = \underline{A}'\underline{\lambda}(k+1) + \underline{Q}\,\underline{\alpha}(k) \qquad (4.5.22)$$

Let

$$\underline{\alpha}(k) \triangleq 2\,\underline{x}(k)$$

Then we have

$$\underline{x}(k+1) = \underline{A}\,\underline{x}(k) - \frac{1}{2}\underline{B}\,\underline{R}^{-1}\underline{B}'\underline{\lambda}(k+1) \quad k=0,\ldots,T-1 \qquad (4.5.23)$$

$$\underline{\lambda}(k) = \underline{A}'\underline{\lambda}(k+1) + 2\,\underline{Q}\,\underline{x}(k) \qquad (4.5.24)$$

$$\underline{x}(0) = \overline{\underline{x}}(0)$$

$$\underline{\lambda}(T) = 2\,\underline{F}\,\underline{x}(T) \qquad (4.5.25)$$

This is the two-point boundary value problem associated with the optimal control problem (4.5.10) and (4.5.11) [A3]. Q.E.D.

Since $\underline{\lambda}(k) = 2\,\underline{K}(k)\underline{x}(k)$ where $\underline{K}(k)$ is the solution of the Riccati equation for the whole system

$$\underline{K}(k) = \underline{Q} + \underline{A}'\,\underline{K}(k+1)[\underline{I} - \underline{B}\,\underline{T}^{-1}(k+1)\underline{B}'\underline{K}(k+1)]\underline{A}$$

$$\underline{K}(T) = \underline{F} \qquad (4.5.26)$$

$$\underline{T}(k+1) = \underline{R} + \underline{B}'\underline{K}(k+1)\underline{B} \qquad (4.5.27)$$

the optimal control $\underline{u}_i^*$ given the optimal coordinating signal $\underline{p}^*(k)$ is

$$\begin{aligned}\underline{u}_i^*(k) &= -\underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1)\underline{A}_{ii}(\hat{\underline{x}}_i(k)-\overline{\underline{x}}_i(k)) - \frac{1}{2}\underline{R}_i^{-1}\underline{B}_i'\underline{\lambda}_i^*(k+1)\\ &= -\underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1)\underline{A}_{ii}(\hat{\underline{x}}_i(k)-\overline{\underline{x}}_i(k)) - \frac{1}{2}[\underline{R}^{-1}\underline{B}'\underline{\lambda}^*(k+1)]_i\end{aligned} \qquad (4.5.28)$$

where $[\underline{a}]_i$ corresponds to the ith component of vector $\underline{a}$.

We now show how $\bar{\underline{x}}_i(k)$ is related to the solution of the deterministic linear regulator problem of the entire system.

Theorem 4.5.2:

Given the optimal coordinating parameters, the unconditional estimates $\bar{\underline{x}}(k)$ of the state of the system by the lower level (given by equation (4.4.35)) are equal to the unconditional estimate of the coordinator, i.e.,

$$\bar{\underline{x}}(k+1) = \underline{A}\,\bar{\underline{x}}(k) - \frac{1}{2}\underline{B}\,\underline{R}^{-1}\underline{B}'\,\underline{\lambda}^*(k+1)$$

$$\bar{\underline{x}}(0) \text{ given} \tag{4.5.29}$$

Proof:

By equation 4.4.30

$$\underline{v}_i^*(0) = -\underline{A}_{ii}\bar{\underline{x}}_i(0) - \underline{K}_i^{-1}(1)\underline{r}_i(1) + \frac{1}{2}(\underline{K}_i^{-1}(1) + \underline{B}_i\underline{R}_i^{-1}\underline{B}_i')\underline{\lambda}_i^*(1) \tag{4.5.30}$$

By equation 4.5.16

$$-\underline{K}_i^{-1}(1)\underline{r}_i(1) + \frac{1}{2}\underline{K}_i^{-1}(1)\underline{\lambda}_i^*(1) - \bar{\underline{x}}_i(1) = \underline{0} \tag{4.5.31}$$

Thus

$$\begin{aligned}\underline{v}_i^*(0) &= -\underline{A}_{ii}\bar{\underline{x}}_i(0) + \underline{x}_i(1) + \frac{1}{2}\underline{B}_i\underline{R}_i^{-1}\underline{B}_i'\underline{\lambda}_i^*(1)\\ &= -\underline{A}_{ii}\bar{\underline{x}}_i(0) + \sum_{j=1}^{N}\underline{A}_{ij}\bar{\underline{x}}_j(0)\\ &= \sum_{j\neq i}\underline{A}_{ij}\bar{\underline{x}}_j(0)\end{aligned} \tag{4.5.32}$$

We then have

$$\bar{\underline{x}}(1) = \underline{A}\,\bar{\underline{x}}(0) - \frac{1}{2}\underline{B}\,\underline{R}^{-1}\underline{B}'\underline{\lambda}^*(1) \tag{4.5.33}$$

By induction, we can easily show that

$$\underline{v}_i^*(k) = \sum_{j\neq i} \underline{A}_{ij}\bar{\underline{x}}_j(k) \tag{4.5.34}$$

and hence equation 4.5.29. Q.E.D.

We have thus verified constraint (4.2.10). Moreover, we have shown that the unconditional mean (a priori estimate) of the ith controller given the optimal coordinating parameter and the uncoupled subsystem is the same as the a priori estimate obtained by the coordinator. The optimal control $\underline{u}_i^*(k)$ is given by

$$\underline{u}_i^*(k) = -\underline{T}_i^{-1}(k+1)\underline{B}_i\underline{K}_i(k+1)\underline{A}_{ii}(\hat{\underline{x}}_i(k)-\bar{\underline{x}}_i(k)) - [\underline{T}^{-1}(k+1)\underline{B}'\underline{K}(k+1)\underline{A}\,\bar{\underline{x}}(k)]_i \tag{4.5.35}$$

where

$$\underline{T}(k+1) = \underline{R} + \underline{B}'\underline{K}(k+1)\underline{B} \tag{4.5.36}$$

This optimal control consists of two parts, a closed loop part which has been discussed before and an open loop part. The open loop part is the optimal deterministic control for the whole system assuming no measurements are made. Thus the optimal control $\underline{u}_i^*(k)$ has a deterministic component which takes into account the effect of the coupling and a closed-loop part which utilizes the local information available. The closed loop part resembles the solution of a tracking problem where the a priori estimate by the coordinator is the reference state.

6. Discussion and Perspectives

We have obtained an off-line decomposition of the linear-quadratic-Gaussian problem. It is found that the optimal control strategy consists of two parts: a closed-loop part which can be generated by the lower level controller himself and an open-loop part which depends on the coordinating parameter $\underline{p}$. The closed-loop part consists of the optimal deterministic gain for the ith subsystem acting on the difference of two estimates. The optimal coordinating parameter $\underline{p}$ is essentially the costate corresponding to the optimal deterministic control of the entire system using the mean of $\underline{x}(0)$ as its initial state. Then the open loop part is the optimal deterministic control of the whole system. The scheme of control is simpler than the solution to the optimal dynamic team since it requires less on-line and off-line computation. Compared with the centralized case, when there is communication among all the controllers, it is also simpler since a full dimensional Kalman-Bucy filter has been replaced by N local filters. The decrease in computation and communication is accompanied by a loss in mathematical optimality.

CHAPTER 5

DECOMPOSITION OF STOCHASTIC DYNAMIC SYSTEMS (ON-LINE)

1. Introduction

In this chapter we study the on-line decomposition of stochastic dynamic systems. The off-line decomposition of stochastic dynamic systems has been considered in the previous two chapters. Based on the a priori information, the coordinator transmits coordinating parameters to the lower level controllers. With the optimal choice of these parameters, the system is coordinated in the sense that the action of the other subsystems on the ith subsystem and its estimate by the ith controller are equal on the average. Once the system starts running, the coordinator's duty is finished.

In some situations, the coordinator receives new information while the system is running. This new information can be used to improve the performance of the system. Instead of off-line coordination, we thus have on-line periodic coordination with the coordinator processing the new information and transmitting new coordinating parameters. Two kinds of on-line coordination will be considered in this chapter, depending on the coordinator's assumption about the availability of future information. When the coordinator assumes that future information will not be available, open-loop feedback optimal coordination is obtained. When future information is assumed to be available, then truly closed-loop coordination is obtained.

Roughly speaking, the issue of periodic on-line coordination can be explained as follows. Each local controller collects all his measurements (e.g. once a day). On the basis of his own measurements, coordinating signals, etc., he makes his (daily) decisions. Every-so-often

(say once a week) each local controller transmits all his measurements to the coordinator. The two cases considered correspond to

(a) the coordinator does not know if and when local measurements will be transmitted, hence he operates under the pessimistic assumption that no further measurements will be made (open loop feedback optimal strategy).

(b) the coordinator knows a priori that he will receive periodically all measurements, and his coordinating strategy reflects the knowledge that in the future he will receive such measurements (the closed loop case).

In the next section we formulate the on-line coordination problem when the coordinator makes measurements after the system starts running. The open loop feedback optimal concept in stochastic control is applied to coordination in Section 3. In Section 4, the open loop feedback optimal coordination of the LQG problem is investigated. The solutions are found to be rather physically intuitive. In Section 5, we study the truly closed loop mode of coordination. A functional equation which has to be solved is obtained. This is compared with the open loop feedback optimal case. In Section 6 a special linear dynamic team problem consisting of independent subsystems with the only coupling in the terminal cost is considered. This is used in Section 7 to obtain a decomposition for the lower level problem between updating times for the closed loop optimal

coordination of the linear-quadratic-Gaussian problem. The resulting control strategies are very similar in form to those obtained from open loop feedback optimal case. The difference between open loop feedback optimal coordination and closed loop optimal coordination is discussed.

2. Formulation of the On-Line Coordination Problem

Let the coordinator collect the measurements of all the lower level controllers periodically every ℓ units of time. For simplicity, we assume $T = m\ell$, where m is some integer.

Let $Y_0(k)$ be the information available to the coordinator at time k. Then

$$Y_0(k\ell) = Y_0(k\ell+1) = \dots = Y_0(k\ell+\ell-1)$$
$$= \{Y_i(k\ell),\ U_i(k\ell-1);\ i=1,\dots,N\}$$
$$k = 0,\dots,m-1 \tag{5.2.1}$$

The control of the ith controller is allowed to depend on $Y_0(k)$ as well as $Y_i(k)$ and $U_i(k-1)$. We shall show that in certain cases $Y_0(k)$ can be replaced by some sufficient statistics. Given his available information $Y_0(k)$, the coordinator requires $\underline{v}_i(t)$ to be equal to $\sum\limits_{j\neq i}^{N} \underline{g}_{ij}(\underline{x}_j(t))$ on the average, where $t \geq k$. Thus we have the following formulation.

Problem 5.1:

Given

$$\underline{x}_i(k+1) = \underline{f}_i(\underline{x}_i(k),\ \underline{v}_i(k),\ \underline{u}_i(k),\ \underline{\xi}_i(k))$$
$$i = 1,\dots,N \tag{5.2.2}$$

$$J = \sum_{i=1}^{N} J_i$$

$$J_i = E\{K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k),\ \underline{u}_i(k))\} \tag{5.2.3}$$

$$E\{\underline{v}_i(t) - \sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(t)) \,|\, Y_0(k)\} = \underline{0} \quad t \geq k$$

$$k = 0,\ldots,T-1$$

$$i = 1,\ldots,N \tag{5.2.4}$$

$$\underline{u}_i(k) = \underline{\gamma}_i^{\,k}(Y_i(k),\, U_i(k-1),\, Y_0(k);\, \tilde{I}) \tag{5.2.5}$$

$$\underline{v}_i(k) = \underline{\eta}_i^{\,k}(Y_0(k);\, \tilde{I}) \tag{5.2.6}$$

where $\tilde{I}$ has the same interpretation as in Section 3.3. Find optimal control strategies $\underline{\gamma}_i^{\,k}$ and $\underline{\eta}_i^{\,k}$, $i = 1,\ldots,N$; $k = 0,\ldots,T-1$ such that J is minimized.

Comparing with the off-line case discussed in Chapter 3, we see that the constraint has to be satisfied more exactly. In the previous case, the constraint is only required to be satisfied with respect to the a priori information. Now it has to be satisfied with respect to the updated information of the coordinator. By the nested property of the conditional expectation, it is easily seen that equation 5.2.4 implies equation 3.3.3 since

$$E\{\underline{v}_i(t) - \sum_{j\neq i}^{N} \underline{g}_{ij}(\underline{x}_j(t))\} = E\{E\{\underline{v}_i(t) - \sum_{j\neq i}^{N} \underline{g}_{ij}(\underline{x}_j(t))\,|\,Y_0(k)\}\}$$

$$t \geq k \tag{6.2.7}$$

Whether it is off-line coordination as discussed in Chapter 3 or on-line coordination treated here, the lower level controls $\underline{u}_i(k)$ are all closed loop, i.e., they depend on the past information available and are computed based on the assumption that further measurements will be made. The terms "off-line" and "on-line" refer only to the mode of coordination. For on-line coordination, there are many possibilities. We shall consider two here, open-loop feedback optimal coordination and closed loop optimal

coordination, and discuss their similarities and differences in both the general, as well as the linear-quadratic-Gaussian case.

3. Open-Loop Feedback Optimal Periodic Coordination

The philosophy of open-loop feedback optimal controls is essentially the following [D2, C5, T2]. At each time k in the control interval

1. The statistics of the state of the system $\underline{x}(k)$ is generated (possibly with a nonlinear filter) from the available observations.

2. Assuming that no measurements will be made in future, the optimal control sequence $\underline{u}^*(k)$, $\underline{u}^*(k+1),\ldots,\underline{u}^*(T-1)$ is generated based on the currently available data by solving an open-loop control problem, the cost functional being the cost to go from time k conditioned on the data available at time k.

3. The optimal control sequence is applied from time k to time k' when additional measurements are made. Steps 1, 2 are then repeated to obtain a new sequence of optimal controls $\underline{u}^*(k'),\ldots,u^*(T-1)$.

The name open-loop feedback optimal is used because essentially an open-loop stochastic control problem is solved at each updating time and then the optimal controls are applied in a feedback form.

Applying this philosophy to the coordination problem, we have the following scheme. At each time $k = 0, \ell, 2\ell,\ldots,T - \ell,$

1. The statistics of the state of the system $\underline{x}(k)$ is generated (possibly with a nonlinear filter) from the available data $Y_0(k)$ by the coordinator.

2. Assuming that no measurements will be made by the coordinator in future, the coordinator then faces a problem similar to Problem 3.1 except that the system starts from time k and the a priori statistics on $\underline{x}(0)$ is replaced by the conditional density of $\underline{x}(k)$ given $Y_0(k)$.

3. The coordinating parameters $\underline{p}^*(k),\ldots,\underline{p}^*(T-1)$ can be found. They would define the lower level problems from which the optimal control strategies $\underline{\gamma}_i^{\,t}$, $\underline{\eta}_i^{\,t}$, $t = k,\ldots,T - 1$, $i = 1,\ldots,N$ are computed by the local level. These optimal control strategies are applied until $t = k + \ell - 1$ when a new set of data $Y_0(k+\ell)$ is available to the coordinator. The whole process is then repeated.

Using this approach, Problem 5.1 can be solved by the following hierarchical scheme.

Lower Level:

$$n\ell \leq k < n\ell + \ell \qquad n = 0,\ldots,m-1$$

$$\underline{x}_i(k+1) = \underline{f}_i(\underline{x}_i(k),\ \underline{v}_i(k),\ \underline{u}_i(k),\ \underline{\xi}_i(k)) \tag{5.3.1}$$

$$\underline{u}_i(k) = \underline{\gamma}_i^{\,k}(n\ell)(Y_i(k),\ U_i(k-1),\ Y_0(n\ell);\ \tilde{I}) \tag{5.3.2}$$

$$\underline{v}_i(k) = \underline{\eta}_i^{\,k}(n\ell)(Y_0(n\ell);\ \tilde{I}) \tag{5.3.3}$$

$$\tilde{J}_i(\underline{p}(n\ell),\ Y_0(n\ell)) = E\{K_i(\underline{x}_i(T)) + \sum_{k=n\ell}^{T-1} L_i(\underline{x}_i(k),\ \underline{u}_i(k)) + \underline{p}_i'(k;n\ell)\underline{v}_i(k) - \sum_{j\neq i} \underline{p}_j'(k;n\ell)\underline{g}_{ji}(\underline{x}_i(k))\,|\,Y_0(n\ell)\} \tag{5.3.4}$$

Find $\underline{\gamma}_i^{\,k}(n\ell)$ and $\underline{\eta}_i^{\,k}(n\ell)$, $k \geq n\ell$, $i = 1,\ldots,N$ such that $\tilde{J}_i(\underline{p}(n\ell),\ Y_0(n\ell))$ is minimized.

$$\underline{p}(n\ell) \triangleq \{\underline{p}(k;n\ell)\}_{k=n\ell}^{T-1} \tag{5.3.5}$$

Let

$$\tilde{J}_i^*(\underline{p}(n\ell),\ Y_0(n\ell)) \triangleq \min_{\underline{\gamma}_i,\underline{\eta}_i} \tilde{J}_i(\underline{p}(n\ell),\ Y_0(n\ell)) \tag{5.3.6}$$

Higher Level:

$$\underset{\underline{p}(n\ell)}{\text{Maximize}} \sum_{i=1}^{N} \tilde{J}_i^*(\underline{p}(n\ell),\ Y_0(n\ell)) \tag{5.3.7}$$

$$\underline{p}(n\ell) = \underline{\beta}(Y_0(n\ell)) \tag{5.3.8}$$

where $\underline{\beta}$ is some measurable function of $Y_0(n\ell)$.

Apart from solving the maximization problem, the coordinator is also responsible for generating the conditional density of $\underline{x}_i(n\ell)$ given $Y_0(n\ell)$ which is necessary for the definition of the lower level problems. In addition to $\underline{p}^*(n\ell)$, this probability has also to be transmitted. For the

lower level controllers, although the optimal $\underline{\gamma}_i^k(n\ell)$ and $\underline{\eta}_i^k(n\ell)$; $k \geq n\ell$ are computed, only $\underline{\gamma}_i^k(n\ell)$ and $\underline{\eta}_i^k(n\ell)$ with $n\ell \leq k < n\ell + \ell$ are used in the actual control.

We notice that the open-loop problem has to be solved at each updating time by the coordinator and then the stochastic control problem solved by the lower level controllers. Depending on the nature of the lower and higher level problems, this open-loop feedback optimal strategy may or may not be feasible. When the updating interval is very long, then the computations involved may become manageable. Again, analytical results can be obtained for the linear-quadratic-Gaussian case. This is discussed in the next section.

4. Open Loop Feedback Optimal Coordination of the Linear-Quadratic-Gaussian Problem

In Chapter 4, we obtained the optimal control strategies for the linear-quadratic-Gaussian case when off-line coordination is assumed. This will be used to obtain the open loop feedback optimal coordination of the linear-quadratic-Gaussian problem.

In Section 4.3, equation (4.3.1) gives

$$\underline{u}_i^*(k) = -\underline{D}_i(k+1)(\hat{\underline{x}}_i(k)-\bar{\underline{x}}_i(k)) - \underline{E}_i(k+1)\underline{p}_i^*(k) \qquad (5.4.1)$$

where $\bar{\underline{x}}_i(k)$ is the estimate generated by the ith controller using his a priori information and the coordinating parameters $\underline{p}^*(k)$'s. $\hat{\underline{x}}_i(k)$ is generated using the measurements in addition. They are given by equations (4.3.7) and (4.3.10).

Using the open loop feedback optimal philosophy, we have the following.

At any time k, let $n\ell$ be the last updating time. Thus

$$n\ell \leq k < n\ell + \ell \qquad (5.4.2)$$

Then

$$\underline{u}_i^*(k) = -\underline{D}_i(k+1)(\hat{\underline{x}}_i(k|k)-\bar{\underline{x}}_i(k|n\ell)) - \underline{E}_i(k+1)\underline{p}_i^*(k;n\ell) \qquad (5.4.3)$$

where the gain matrices are the same as those given in Chapter 4.

$$\hat{\underline{x}}_i(k|k) \triangleq E\{\underline{x}_i(k)|\ Y_i(k),\ U_i(k-1);\ I_i,\ \underline{p}^*(n\ell),\ \bar{\underline{x}}_i(n\ell|n\ell),\underline{\Sigma}_{ii}(n\ell|n\ell)\} \qquad (5.4.4)$$

$$\bar{\underline{x}}_i(k|n\ell) \triangleq E\{\underline{x}_i(k)|I_i,\ \underline{p}^*(n\ell),\ \bar{\underline{x}}_i(n\ell|n\ell)\} \qquad (5.4.5)$$

I_i is the decentralized a priori information of the ith controller about his subsystem. $\bar{\underline{x}}_i(k|n\ell)$ is the estimate generated by the ith

controller based on the coordinating parameters $\underline{p}^*(n\ell)$ and $\bar{\underline{x}}_i(n\ell|n\ell)$. $\hat{\underline{x}}_i(k|k)$ is the decentralized estimate generated by the ith controller using the coordinating parameters $\underline{p}^*(n\ell)$, the statistics $\bar{\underline{x}}_i(n\ell|n\ell)$ and $\underline{\Sigma}_{ii}(n\ell|n\ell)$, and the data $Y_i(k)$, $U_i(k-1)$.

The state estimate of the coordinator is generated as follows.

$$\bar{\underline{x}}(k+1|k+1) = \underline{A}\,\bar{\underline{x}}(k|k) + \underline{B}\,\underline{u}(k) + \underline{G}(k+1)[\underline{y}(k+1)-\underline{C}(\underline{A}\,\bar{\underline{x}}(k|k)+\underline{B}\,\underline{u}(k)]$$

$$\bar{\underline{x}}(0|0) = \bar{\underline{x}}(0) \tag{5.4.6}$$

$$\underline{G}(k+1) = \underline{\Sigma}(k+1|k+1)\underline{C}'\underline{\Theta}^{-1}(k+1) \tag{5.4.7}$$

$$\underline{\Sigma}(k+1|k+1) = [\underline{C}'\underline{\Theta}^{-1}(k+1)\underline{C} + (\underline{A}\,\underline{\Sigma}(k|k)\underline{A}' + \underline{\Xi}(k))^{-1}]^{-1}$$

$$\underline{\Sigma}(0|0) = \underline{\Sigma}(0) - \underline{\Sigma}(0)\,\underline{C}'(\underline{C}\,\underline{\Sigma}(0)\underline{C}' + \underline{\Theta}(0))^{-1}\underline{C}\,\underline{\Sigma}(0) \tag{5.4.8}$$

The coordinating parameters $\underline{p}^*(k;n\ell)$ are given by

$$\underline{p}^*(k;n\ell) = -\underline{\lambda}^*(k+1;n\ell) = -2\,\underline{K}(k+1)\,\bar{\underline{x}}(k+1|n\ell) \tag{5.4.9}$$

where

$$\bar{\underline{x}}(k+1|n\ell) = (\underline{I} - \underline{B}\,\underline{T}^{-1}(k+1)\underline{B}'\underline{K}(k+1))\underline{A}\,\bar{\underline{x}}(k|n\ell) \tag{5.4.10}$$

$\bar{\underline{x}}(n\ell|n\ell)$ is given by (5.4.6).

The decentralized state estimates $\hat{\underline{x}}_i(k|k)$ and $\bar{\underline{x}}_i(k|n\ell)$ are found as follows.

$$\hat{\underline{x}}_i(k+1|k+1) = \underline{A}_{ii}\hat{\underline{x}}_i(k|k) + \underline{v}_i^*(k|n\ell) + \underline{B}_i\underline{u}_i(k) + \underline{G}_i(k+1|n\ell)[\underline{y}_i(k+1)$$

$$-\underline{C}_i(\underline{A}_{ii}\hat{\underline{x}}_i(k|k) + \underline{v}_i^*(k|n\ell) + \underline{B}_i\underline{u}_i(k)] \tag{5.4.11}$$

$$\hat{\underline{x}}_i(n\ell|n\ell) = \bar{\underline{x}}_i(n\ell|n\ell) \tag{5.4.12}$$

$$\underline{G}_i(k+1|n\ell) = \underline{\Sigma}_i(k+1|k+1;n\ell)\underline{C}_i'\underline{\Theta}_i^{-1}(k+1) \tag{5.4.13}$$

$$\underline{\Sigma}_i(k+1|k+1;n\ell) = [\underline{C}_i'\underline{\Theta}_i^{-1}(k+1)\underline{C}_i + (\underline{A}_{ii}\underline{\Sigma}_i(k|k;n\ell)\underline{A}_{ii}' + \underline{\Xi}_i(k))^{-1}]^{-1} \tag{5.4.14}$$

$$\underline{\Sigma}_i(n\ell|n\ell;n\ell) = \underline{\Sigma}_{ii}(n\ell|n\ell) \tag{5.4.15}$$

$$\overline{\underline{x}}_i(k+1|n\ell) = -\underline{K}_i^{-1}(k+1)[\underline{r}_i(k+1|n\ell) + \frac{1}{2}\underline{p}_i^*(k;n\ell)] \tag{5.4.16}$$

$\underline{v}_i^*(k|n\ell)$ and $\underline{r}_i(k|n\ell)$ are given by the following equations.

$$\begin{aligned}\underline{v}_i^*(n\ell|n\ell) = &-\underline{A}_{ii}\overline{\underline{x}}_i(n\ell|n\ell) - \underline{K}_i^{-1}(n\ell+1)\underline{r}_i(n\ell+1|n\ell)\\ &-\frac{1}{2}\underline{S}_i^{-1}(n\ell+1)\underline{p}_i^*(n\ell;n\ell)\end{aligned} \tag{5.4.17}$$

$$\begin{aligned}\underline{v}_i^*(k|n\ell) = &\underline{A}_{ii}\underline{K}_i^{-1}(k)[\underline{r}_i(k|n\ell) + \frac{1}{2}\underline{p}_i^*(k-1;n\ell)]\\ &-\underline{K}_i^{-1}(k+1)\underline{r}_i(k+1|n\ell) - \frac{1}{2}\underline{S}_i^{-1}(k+1)\underline{p}_i^*(k;n\ell)\\ &n\ell < k < n\ell + \ell\end{aligned} \tag{5.4.18}$$

$$\begin{aligned}\underline{r}_i(n\ell|n\ell) = &-\frac{1}{2}\tilde{\underline{p}}_i^*(n\ell;n\ell) - \frac{1}{2}\underline{A}_{ii}'\underline{p}_i^*(n\ell;n\ell)\\ &-\underline{A}_{ii}'\underline{S}_i(n\ell+1)\underline{A}_{ii}\overline{\underline{x}}_i(n\ell|n\ell)\end{aligned} \tag{5.4.19}$$

$$\begin{aligned}\underline{Q}_i\underline{K}_i^{-1}(k)\underline{r}_i(k|n\ell) = &-\frac{1}{2}\tilde{\underline{p}}_i^*(k;n\ell) - \frac{1}{2}\underline{A}_{ii}'\underline{p}_i^*(k;n\ell)\\ &+\frac{1}{2}\underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii}\underline{K}_i^{-1}(k)\underline{p}_i^*(k+1;n\ell)\\ &n\ell < k \leq n\ell + \ell\end{aligned} \tag{5.4.20}$$

$$\underline{r}_i(T|n\ell) = \underline{0} \tag{5.4.21}$$

Essentially at each updating time $n\ell$, the coordinator evaluates a state estimate $\overline{\underline{x}}(n\ell|n\ell)$ and new covariance $\underline{\Sigma}(n\ell|n\ell)$ using his measurements. From these the optimal coordinating parameters $\underline{p}^*(k;n\ell)$ are computed. $\overline{\underline{x}}_i(n\ell|n\ell)$, $\underline{\Sigma}_{ii}(n\ell|n\ell)$ and $\underline{p}^*(k;n\ell)$, $k = n$,....,$n\ell+\ell-1$, are transmitted

to the ith lower level controller.

The ith lower level controller generates the new a priori estimate $\bar{\underline{x}}_i(k|n\ell)$ from the coordinating parameters as well as the estimate $\underline{x}_i(k|k)$ from his own data. From Theorem 4.5.2 we can see that $\bar{\underline{x}}_i(k|n\ell)$ is the same as that generated by the coordinator using equation (5.4.10). The optimal control strategy given by equation (5.4.1) consists of a part depending on the difference between these two estimates and a part specified by the coordinator through the coordinating parameters. When these two estimates are the same, essentially the coordinator takes over the control of the system. This can happen when the coordinator updates his information as often as the lower level controllers. In this case from Appendix D,

$$\begin{aligned} \underline{u}_i^*(k) &= -\underline{E}_i(k+1)\underline{p}_i^*(k;k) \\ &= -[\underline{T}^{-1}(k+1)\underline{B}'\underline{K}(k+1)\underline{A}\,\bar{\underline{x}}(k|k)]_i \end{aligned} \tag{5.4.22}$$

Thus the control strategy approaches that given by a separation theorem asymptotically.

<u>Remark</u>:

The linear-quadratic-Gaussian problem is really a very special case. We note that for open loop feedback optimal coordination, the lower level controllers do not need the whole set of coordinating parameters from time $n\ell$ to T-1. In general, this may not be the case. Recall also that for centralized information structure, there is no difference between the open loop feedback optimal control and the closed loop optimal control for the LQG problem. In the case of coordination, however, there will be a difference, as can be seen from the next section. The fact that the

coordinator assumes the availability of future information introduces some features which are not present in the open loop feedback optimal case.

5. Closed-Loop Optimal Periodic Coordination

In closed-loop coordination, the coordinator chooses his coordinating parameters knowing that measurements will be made in the future. This has both advantages and disadvantages. Instead of having a different set of coordinating parameters which are changed at each updating time, only one set of coordinating parameters will be required but the future ones depend on measurements still not yet available to the coordinator. Complete decoupling of the lower level problems is not obtained in this case, however. The following theorem uses the closed loop nature of the coordination to reduce Problem 5.1 to another problem.

Theorem 5.5.1:

If closed-loop periodic coordination is used and the coordinator has perfect memory, the Problem 5.1 is equivalent to the following problem.

Problem 5.2:

Given

$$\underline{x}_i(k+1) = \underline{f}_i(\underline{x}_i(k), \underline{v}_i(k), \underline{u}_i(k), \underline{\xi}_i(k)) \qquad i = 1,\ldots,N \tag{5.5.1}$$

$$J = \sum_{i=1}^{N} J_i \tag{5.5.2}$$

$$J_i = E\{K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k), \underline{u}_i(k))\} \tag{5.5.3}$$

$$E\{\underline{v}_i(t) - \sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(t)) | Y_0(k)\} = \underline{0}$$

$$k + \ell > t \geq k \qquad k = 0, \ell, \ldots, T - \ell$$

$$i = 1,\ldots,N \tag{5.5.4}$$

$$\underline{u}_i(k) = \underline{\gamma}_i^{\,k}(Y_i(k),\ U_i(k-1),\ Y_0(k);\ \tilde{I}) \tag{5.5.5}$$

$$\underline{v}_i(k) = \underline{\gamma}_i^{\,k}(Y_0(k);\ \tilde{I}) \tag{5.5.6}$$

Find optimal $\underline{\gamma}_i^{\,k}$ and $\underline{\eta}_i^{\,k}$, $i = 1,\ldots,N$; $k = 0,\ldots,T-1$ such that J is minimized.

Proof:

We need only to show that constraint (5.2.4) is equivalent to constraint (5.5.4).

Equation (5.2.4) obviously implies equation (5.5.4).

Suppose

$$E\{\underline{v}_i(t) - \sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(t))\,|\,Y_0(k)\} = \underline{0}$$

for all $k + \ell > t \geq k$; $k = 0, \ell,\ldots,T-\ell$.

Then consider

$$E\{\underline{v}_i(t) - \sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(t))\,|\,Y_0(k)\}$$

where $t \geq k + \ell$.

Since measurements will be made in future and closed loop coordination is used, there exists an updating time k' such that $k' + \ell > t \geq k'$ and

$$E\{\underline{v}_i(t) - \sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(t))\,|\,Y_0(k')\} = \underline{0} \tag{5.5.7}$$

Thus

$$E\{\underline{v}_i(t) - \sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(t))\,|\,Y_0(k)\} = E\{E\{\underline{v}_i(t) - \sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(t))\,|\,Y_0(k')\}\,|\,Y_0(k)\} = \underline{0} \tag{5.5.8}$$

The first equality follows from the perfect memory of the coordinator and the nested property of the conditional expectation. Q.E.D.

In Section 3.4 we considered a constrained stochastic control problem and used it to obtain an off-line decomposition for coupled dynamic systems. There the constraint is with respect to the unconditional mean and hence the decomposition is off-line. To obtain an on-line decomposition which is closed loop optimal, we have to consider constraints which are conditioned on on-line measurements. Corresponding to Problem 3.3 we thus have the following problem.

Problem 5.3

System: $$\underline{x}(k+1) = \underline{f}(\underline{x}(k), \underline{v}(k), \underline{u}(k), \underline{\xi}(k)) \tag{5.5.9}$$

$$E\{\underline{H}(\underline{x}(t), \underline{v}(t)) | Y_0(k)\} = \underline{0}$$

$$k + \ell > t \geq k \ , \ k = 0, \ell, 2\ell, \ldots \tag{5.5.10}$$

Measurement: $$\underline{y}(k) = \underline{h}(\underline{x}(k), \underline{\theta}(k)) \tag{5.5.11}$$

Cost Functional: $$J = E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k), \underline{u}(k))\} \tag{5.5.12}$$

$\underline{\xi}(k)$, $\underline{\theta}(k)$, $k = 0,\ldots,T-1$ and $\underline{x}(0)$ are random vectors with known statistics.

$$Y(k) = \{\underline{y}(0),\ldots,\underline{y}(k); \underline{u}(0),\ldots,\underline{u}(k-1)\} \tag{5.5.13}$$

$$Y_0(k\ell) = Y_0(k\ell+1) = \ldots = Y_0(k\ell+\ell-1) = Y(k\ell)$$

$$k = 0,\ldots,m-1 \tag{5.5.14}$$

$$\underline{u}(k) = \underline{\gamma}^k(Y(k)) \tag{5.5.15}$$

$$\underline{v}(k) = \underline{\eta}^k(Y_0(k)) \tag{5.5.16}$$

It is required to find $\underline{\gamma}^*$ and $\underline{\eta}^*$ such that J is minimized and the constraint is satisfied.

Corresponding to Theorem 3.4.1 we have the following.

<u>Theorem 5.5.2</u>:

Let $\underset{\substack{a\\b}}{\mathrm{VSP}}\{L(a,b)\}$ denote the value of the saddle-point of L where a is the minimizing variable and b is the maximizing variable. If the saddle point exists in the following functional equation, then the optimal cost for Problem 5.4 is given by

$$E\{V(Y_0(0),0)\} \tag{5.5.17}$$

where $V(Y_0(k),k)$ satisfies the functional equation

$$V(Y_0(k),k) = \underset{\substack{\underline{\gamma}^k,\ldots,\underline{\gamma}^{k+\ell-1};\ \underline{v}(k),\ldots,v(k+\ell-1)\\ \underline{p}(k),\ldots,\underline{p}(k+\ell-1)}}{\mathrm{VSP}} E\{\sum_{t=k}^{k+\ell-1} L(\underline{x}(t),\ \underline{u}(t))$$

$$+ \underline{p}'(t)\underline{H}(\underline{x}(t),\ \underline{v}(t)) + V(Y_0(k+\ell),\ k+\ell)\,|\,Y_0(k)\}$$

$$k = 0,\ \ell,\ldots,T-\ell \tag{5.5.18}$$

$$V(Y_0(T),T) = E\{K(\underline{x}(T))\,|\,Y_0(T)\} \tag{5.5.19}$$

<u>Proof</u>:

Define

$$V(Y_0(k),k) = \underset{\substack{\underline{\gamma}^k,\ldots,\underline{\gamma}^{T-1}\\ \underline{v}(t)\ \varepsilon\ \Omega_1(Y_0(t),t),\ t=k,\ldots,k+\ell-1\\ \underline{\eta}^t\ \varepsilon\ \Omega_2(Y_0(t),t),\ t=k+\ell,\ldots,T-1}}{\mathrm{Min}} E\{\sum_{t=k}^{T-1} L(\underline{x}(t),\underline{u}(t))+K(\underline{x}(T))\,|\,Y_0(k)\}$$

$$= \underset{\substack{\underline{\gamma}^k,\ldots,\underline{\gamma}^{k+\ell-1}\\ \underline{v}(t)\ \varepsilon\ \Omega_1(Y_0(t),t),\ t=k,\ldots,k+\ell-1}}{\mathrm{Min}} [E\{\sum_{t=k}^{k+\ell-1} L(\underline{x}(t),\underline{u}(t))\,|\,Y_0(k)\}$$

$$+ \min_{\substack{\underline{\gamma}^{k+\ell},\ldots,\underline{\gamma}^{T-1} \\ \underline{\eta}^t \in \Omega_2(Y_0(t),t),\ t=k+\ell,\ldots,T-1}} E\{\sum_{t=k+\ell}^{T-1} L(\underline{x}(t),\underline{u}(t))+K(\underline{x}(T))\,|\,Y_0(k)\}] \tag{5.5.20}$$

$$\Omega_1(Y_0(t),t) = \{\underline{v}\,|\,E\{\underline{H}(\underline{x}(t),\underline{v})\,|\,Y_0(t)\} = \underline{0}\} \tag{5.5.21}$$

$$\Omega_2(Y_0(t),t) = \{\underline{\eta}\,|\,E\{\underline{H}(\underline{x}(t),\underline{\eta}(Y_0(t))\,|\,Y_0(t)\} = \underline{0}\} \tag{5.5.22}$$

From Appendix A, the second term on the right hand side of (5.5.20) to be minimized is $E\{V(Y_0(k+\ell),k+\ell)\,|\,Y_0(k)\}$. Thus

$$V(Y_0(k),k) = \min_{\substack{\underline{\gamma}^{k},\ldots,\underline{\gamma}^{k+\ell-1} \\ \underline{v}(t) \in \Omega_1(Y_0(t),t),\ t=k,\ldots,k+\ell-1}} E\{\sum_{t=k}^{k+\ell-1} L(\underline{x}(t),\underline{u}(t)) + V(Y_0(k+\ell),k+\ell)\,|\,Y_0(k)\} \tag{5.5.23}$$

Form the Lagrangian for equation (5.5.23) as

$$E\{\sum_{t=k}^{k+\ell-1} L(\underline{x}(t),\underline{u}(t)) + \underline{p}'(t)\underline{H}(\underline{x}(t),\underline{v}(t)) + V(Y_0(k+\ell),k+\ell)\,|\,Y_0(k)\} \tag{5.5.24}$$

If a saddle point exists, then $V(Y_0(k),k)$ is given by equation (5.5.18). By the same argument as in Theorem 4.3.1,

$$V(Y_0(0),0) = \min_{\substack{\underline{\gamma}^{k},\ k=0,\ldots,T-1 \\ \underline{\eta}^k \in \Omega_2(Y_0(k),k),\ k=0,\ldots,T-1}} E\{\sum_{k=0}^{T-1} L(\underline{x}(k),\underline{u}(k)) + K(\underline{x}(T))\} \tag{5.5.25}$$

It is thus the optimal cost for Problem 5.3. Q.E.D.

By the property of the saddle point, the following corollary is obvious.

Corollary 5.5.3:

If the optimal Lagrange multipliers $\underline{p}^*(k) = \underline{\beta}^k(Y_0(k))$ are known, then

$$V(Y_0(k),k) = \min_{\substack{\Upsilon^k,\ldots,\Upsilon^{k+\ell-1} \\ \underline{v}(k),\ldots,\underline{v}(k+\ell-1)}} E\{\sum_{t=k}^{k+\ell-1} L(\underline{x}(t),\underline{u}(t)) + \underline{p}^{*\prime}(t)\underline{H}(\underline{x}(t),\underline{v}(t))$$

$$+ V(Y_0(k+\ell),k+\ell)\,|\,Y_0(k)\} \quad k = 0, \ell,\ldots,T-\ell \tag{5.5.26}$$

$$V(Y_0(T),T) = E\{K(\underline{x}(T))\,|\,Y_0(T)\} \tag{5.5.27}$$

Remark:

It is also possible to substitute the VSP operation in Theorem 5.5.2 with maxmin or minmax.

If we apply the corollary to Problem 5.2, we will find that the terms involving $L(\underline{x}(t),\underline{u}(t))$ and $\underline{p}^{*\prime}(t)\underline{H}(\underline{x}(t),\underline{v}(t))$ are separable into the subsystems. However, $V(Y_0(k+\ell),k+\ell)$ cannot be separated into a sum of N independent parts. Thus, although we have an optimal stochastic control problem consisting of N uncoupled subsystems, the cost functional, which is essentially separable has a terminal term which is not separable. This makes our problem of finding the optimal controls considerably harder than the open loop feedback optimal case. The main difference between the two cases lies in the assumption of future measurements by the coordinator. When future measurements are assumed to be made, this fact will be made use of by the lower level controllers as well as by the coordinator.

Although (5.5.26) is not as simple as the lower level problem in Section 3, it is simpler than the original problem with decentralized a posteriori information and common a priori information. Moreover, only

one sequence of $\underline{p}$'s need to be chosen by the coordinator. In the open loop feedback optimal case, a different sequence of $\underline{p}$'s have to be chosen at each updating time. This equation is quite nontrivial to solve. In the next two sections, we shall show how this equation can be solved for the linear-quadratic-Gaussian case.

6. A Special Linear Dynamic Team Problem

In this section we shall consider a special linear dynamic team. The solution of this team problem will be used in the next section to obtain the closed loop optimal coordination of the linear-quadratic-Gaussian problem. Problem 5.4 stated below is the team decision problem under consideration. We shall relate its solution to the solutions of Problems 5.5 and 5.6 which are simpler.

Problem 5.4:

$$\underline{x}_i(k+1) = \underline{A}_{ii}\underline{x}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k) + \underline{\xi}_i(k) \tag{5.6.1}$$

$$J = E\Big\{\sum_{i=1}^{N}\sum_{k=0}^{T-1} \underline{x}_i'(k)\underline{Q}_i\underline{x}_i(k) + \underline{u}_i'(k)\underline{R}_i\underline{u}_i(k) + \underline{p}_i'(k)\underline{v}_i(k)$$

$$- \tilde{\underline{p}}_i'(k)\underline{x}_i(k) \quad + \underline{x}'(T)\underline{K}(T)\underline{x}(T)\Big\} \tag{5.6.2}$$

$$\underline{u}_i(k) = \underline{\gamma}_i^{k}(Y_i(k),\ U_i(k-1)) \tag{5.6.3}$$

$\underline{v}_i(k)$ depends only on the a priori information.

$\underline{x}_i(0)$ given $i=1,\dots,N$.

Find $\underline{\gamma}_i$, $\underline{v}_i$, $i=1,\dots,N$ such that J is minimized.

Remark: This is a special case of a dynamic team. The dynamics of the subsystems are uncoupled. The cost functional is essentially uncoupled except for the terminal cost.

Problem 5.5: $i=1,\dots,N$

$$\underline{x}_i(k+1) = A_{ii}\underline{x}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k) + \underline{\xi}_i(k) \tag{5.6.4}$$

$$J_i = E\Big\{\sum_{k=0}^{T-1} \underline{x}_i'(k)\underline{Q}_i\underline{x}_i(k) + \underline{u}_i'(k)\underline{R}_i\underline{u}_i(k) + \underline{p}_i'(k)\underline{v}_i(k)$$

$$- \tilde{\underline{p}}_i'(k)\underline{x}_i(k) + \underline{x}_i'(T)\underline{K}_{ii}(T)\underline{x}_i(T) + 2r_i'(T)\underline{x}_i(T)\Big\}$$

(5.6.5)

$$\underline{u}_i(k) = \underline{\gamma}_i^k(Y_i(k),\ U_i(k-1)) \qquad (5.6.6)$$

$\underline{v}_i(k)$ depends only on the a priori information

$\underline{x}_i(0)$ given

Find $\underline{\gamma}_i^*$, $\underline{v}_i^*$ such that J_i is minimized, $i=1,\ldots,N$.

$\underline{r}_i(T)$ is a n_i dimensional vector.

<u>Remark</u>: Given $\underline{r}_i(T)$, we have N uncoupled stochastic control problems with linear terms in the terminal costs.

<u>Problem 5.6</u> : Same as Problem 5.5.

Find $\underline{\gamma}_i^*$, $\underline{v}_i^*$ and $\underline{r}_i^*(T)$ such that J_i given $\underline{r}_i^*$ is minimized, $i=1,\ldots,N$

and

$$\underline{r}_i^*(T) = \sum_{j\neq i} \underline{K}_{ij}(T)\bar{\underline{x}}_j^*(T) = \sum_{j\neq i} \underline{K}_{ij}(T)E\{\underline{x}_j^*(T)\} \qquad (5.6.7)$$

where $\underline{x}^*(T)$ is the resultant optimal trajectory.

<u>Remark</u>: The solution of Problem 5.6 gives the <u>person-by-person optimal</u> (pbpo) strategy of the team. Given a team with payoff function $J(\underline{\gamma}_1,\ldots,\underline{\gamma}_N)$ the pbpo strategy $(\underline{\gamma}_1^*,\ldots,\underline{\gamma}_N^*)$ is defined by

$$J(\underline{\gamma}_1^*,\ldots,\underline{\gamma}_i^*,\ldots,\underline{\gamma}_N^*) \leq J(\underline{\gamma}_1^*,\ldots,\underline{\gamma}_{i-1}^*,\underline{\gamma}_i,\underline{\gamma}_{i+1}^*,\ldots,\underline{\gamma}_N^*)$$

(5.6.8)

for all i and $\underline{\gamma}_i$.

<u>Theorem 5.6.1</u>: $\underline{\gamma}_i^*$, $\underline{v}_i^*$, $i=1,\ldots,N$ solve Problem 5.4 if and only if they solve Problem 5.6.

<u>Proof</u>:

Necessity:

Suppose $\underline{\gamma}_i^*$, $\underline{v}_i^*$, $i=1,\ldots,N$ solve Problem 5.4. Then in particular

$$E\left\{\sum_{i=1}^{N}\sum_{k=0}^{T-1} \underline{x}_i^{*\prime}(k)\underline{Q}_i\underline{x}_i^*(k) + \underline{u}_i^{*\prime}(k)\underline{R}_i\underline{u}_i^*(k) + \underline{p}_i'(k)\underline{v}_i^*(k) - \tilde{\underline{p}}_i'(k)\underline{x}_i^*(k) \quad + \underline{x}^{*\prime}(T)\underline{K}(T)\underline{x}^*(T)\right\}$$

$$\leq E\left\{\sum_{k=0}^{T-1} \underline{x}_i'(k)\underline{Q}_i\underline{x}_i(k) + \underline{u}_i'(k)\underline{R}_i\underline{u}_i(k) + \underline{p}_i'(k)\underline{v}_i(k) - \tilde{\underline{p}}'(k)\underline{x}_i(k) + 2\left(\sum_{j\neq i}\underline{K}_{ij}(T)\underline{x}_j^*(T)\right)'\underline{x}_i(T) + \underline{x}_i'(T)\underline{K}_{ii}(T)\underline{x}_i(T)\right\}$$

$$+ E\left\{\sum_{j\neq i}\sum_{k=0}^{T-1} \underline{x}_j^{*\prime}(k)\underline{Q}_j\underline{x}_j^*(k) + \underline{u}_j^{*\prime}(k)\underline{R}_j\underline{u}_j^*(k) + \underline{p}_j'(k)\underline{v}_j^*(k) - \tilde{\underline{p}}_j^{*\prime}(k)\underline{x}_j^*(k) + \text{terms independent of } \underline{x}_i(T)\right\} \tag{5.6.9}$$

Thus, by defining $\underline{r}_i^*(T) = \sum_{j\neq i}\underline{K}_{ij}(T)\overline{\underline{x}}_j^*(T)$ and subtracting terms independent of the i^{th} subsystem from each side of equation (5.6.9) we see that Problem 5.6 is solved.

Sufficiency: Using the results in [H3] we can reduce Problem 5.4 to a static linear team with a quadratic payoff. Reference [R1] then shows that the person-by-person optimal strategy is also the optimal strategy for the entire team. Q.E.D.

We shall now solve Problem 5.5.

<u>Theorem 5.6.2</u>: The solution to Problem 5.5 is given by

$$\mathbf{u}_i^*(k) = -\mathbf{D}_i(k+1)(\hat{\mathbf{x}}_i(k) - \bar{\mathbf{x}}_i(k)) - \mathbf{E}_i(k+1)\mathbf{p}_i(k),\ k=0,\ldots,T-1 \tag{5.6.10}$$

$$\mathbf{v}_i^*(0) = -\mathbf{A}_{ii}\bar{\mathbf{x}}_i(0) - \mathbf{K}_i^{-1}(1)\mathbf{r}_i(1) - \frac{1}{2}\mathbf{S}_i^{-1}(1)\mathbf{p}_i(0) \tag{5.6.11}$$

$$\mathbf{v}_i^*(k) = \mathbf{A}_{ii}\mathbf{K}_i^{-1}(k)[\mathbf{r}_i(k) + \frac{1}{2}\mathbf{p}_i(k-1)] - \mathbf{K}_i^{-1}(k+1)\mathbf{r}_i(k+1)$$
$$- \frac{1}{2}\mathbf{S}_i^{-1}(k+1)\mathbf{p}_i(k) \qquad k=1,\ldots,T-1 \tag{5.6.12}$$

where $\mathbf{D}_i$, $\mathbf{E}_i$, $\hat{\mathbf{x}}_i(k)$, $\bar{\mathbf{x}}_i(k)$ are the same as those in Theorem 4.4.2 with $\mathbf{K}_i(T) = \mathbf{K}_{ii}(T)$ and $\mathbf{r}_i(k)$ is given by

$$\mathbf{r}_i(0) = -\frac{1}{2}\tilde{\mathbf{p}}_i(0) - \frac{1}{2}\mathbf{A}_{ii}'\mathbf{p}_i(0) - \mathbf{A}_{ii}'\mathbf{S}_i(1)\mathbf{A}_{ii}\bar{\mathbf{x}}_i(0) \tag{5.6.13}$$

$$\mathbf{Q}_i\mathbf{K}_i^{-1}(k)\mathbf{r}_i(k) = -\frac{1}{2}\tilde{\mathbf{p}}_i(k) - \frac{1}{2}\mathbf{A}_{ii}'\mathbf{p}_i(k) + \frac{1}{2}\mathbf{A}_{ii}'\mathbf{S}_i(k+1)\mathbf{A}_{ii}\mathbf{K}_i^{-1}(k)\mathbf{p}_i(k+1)$$
$$k=1,\ldots,T-1$$

$$\mathbf{r}_i(T) \text{ given} \tag{5.6.14}$$

Moreover, the optimal cost is

$$\bar{\mathbf{x}}_i'(0)\mathbf{K}_i(0)\bar{\mathbf{x}}_i(0) + 2\mathbf{r}_i'(0)\bar{\mathbf{x}}_i(0) + s_i(0) \tag{5.6.15}$$

where $s_i(k)$ is given by equation (4.4.19)

<u>Proof</u>: Since Problem 5.5 and the lower level problem in Chapter 4 (Problem 4.2) differ only in the terminal cost, the functional equation (4.4.22) can also be used here to find the optimal control strategies. The terminal condition, however, has to be modified, with $\mathbf{r}_i(T) \neq \mathbf{0}$.

Q.E.D.

Problem 5.4 can now be solved with the help of Theorem 5.6.1 and Theorem 5.6.2.

Theorem 5.6.3: The control strategies in Theorem 5.6.2 solve Problem 5.4, with

$$\underline{r}(T) = -\frac{1}{2}\underline{p}(T-1) + \frac{1}{2}\underline{\tilde{K}}(T)\underline{K}^{-1}(T)\underline{p}(T-1) \tag{5.6.16}$$

Proof: From equation (4.4.37)

$$\underline{\bar{x}}_i(T) = -\underline{K}_{ii}^{-1}(T)[\underline{r}_i(T) + \frac{1}{2}\underline{p}_i(T-1)] \tag{5.6.17}$$

If

$$\underline{r}_i(T) = \sum_{j\neq i} \underline{K}_{ij}(T)\underline{\bar{x}}_j(T) \tag{5.6.18}$$

then Problem 5.6 and Problem 5.4 have the same solution.

Define

$$\underline{\tilde{K}}(T) = \begin{bmatrix} \underline{K}_{11}(T) & \underline{0} & \cdots & \cdots \\ \underline{0} & \underline{K}_{22}(T) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \underline{K}_{NN}(T) \end{bmatrix} \tag{5.6.19}$$

Then equation (5.6.17) and (5.6.18) give

$$\underline{r}(T) = -(\underline{K}(T) - \underline{\tilde{K}}(T))\,\underline{\tilde{K}}^{-1}(T)\,[\underline{r}(T) + \frac{1}{2}\underline{p}(T)]$$

or

$$\underline{r}(T) = -\frac{1}{2}\underline{p}(T-1) + \frac{1}{2}\underline{\tilde{K}}(T)\underline{K}^{-1}(T)\underline{p}(T-1) \tag{5.6.20}$$

Q.E.D.

The special dynamic team given by Problem 5.4 has been considered in detail. If the $\underline{r}_i(T)$'s are chosen appropriately as given by equation (5.6.16), then the optimal control strategies can be

found in a decentralized manner by finding the solution to Problem 5.5. We may regard the $\underline{r}_i(T)$'s as the coordinating parameters for decomposing the terminal cost.

7. Closer Loop Optimal Coordination of the Linear-Quadratic-Gaussian Problem

In this section we shall look at the closed loop coordination of the linear-quadratic-Gaussian case. Using the results of the previous section, we shall show that although the functional equation (5.5.18) is not separable into the individual subsystems, we can, by introducing an extra coordinating parameter, define optimization problems for each of the lower level controllers. By solving these problems, it is found that closed loop optimal coordination and open loop feedback optimal coordination give rise to essentially the same control strategies for the lower level controllers. However, the gain matrices in the control strategies are different.

Substituting in the right functions, equation (5.5.18) becomes the following:

$$
\begin{aligned}
V(Y_0(k),k) = \max_{\underline{p}(k)\cdots\underline{p}(k+\ell-1)}\ \min_{\substack{\gamma^k,\ldots\gamma^{k+\ell-1}\\ \underline{v}(k),\ldots,\underline{v}(k+\ell-1)}} E\Big\{ &\sum_{i=1}^{N}\sum_{t=k}^{k+\ell-1} \underline{x}_i'(t)\underline{Q}_i\underline{x}_i(t) \\
&+ \underline{u}_i'(t)\underline{R}_i\underline{u}_i(t) + p_i'(t)\underline{v}_i(t) - \tilde{p}_i'(t)\underline{x}_i(t) \\
&+ V(Y_0(k+\ell),k+\ell)\,\big|\,Y_0(k)\Big\} \qquad k = 0,\ell,\ldots, T-\ell
\end{aligned}
\tag{5.7.1}
$$

$$
V(V_0(T),T) = E\{\underline{x}'(T)F\underline{x}(T)\,|\,Y_0(T)\} \tag{5.7.2}
$$

We shall show that at each updating time $k = 0, \ell, \ldots, T-\ell$, the optimal cost to go as evaluated by the coordinator is given by

$$V(Y_0(k), k) = \bar{\underline{x}}'(k|k)\underline{K}(k)\bar{\underline{x}}(k|k) + b(k) \tag{5.7.3}$$

where $\bar{\underline{x}}(k|k)$ is the estimate of the state of the system by the coordinator given by equation (5.4.6)

$\underline{K}(k)$ is the solution to the Riccati equation for the entire system.

$b(k)$ is a precomputable scalar.

This will be proved by induction. First, we shall need the following lemma.

Lemma 5.7.1: Let $\bar{\underline{x}}(k|k)$ and $\underline{\Sigma}(k|k)$ be as defined in Section 4. Then for any positive definite matrix $\underline{M}$, and $k > m$,

$$E\{\bar{\underline{x}}'(k|k)\underline{M}\,\bar{\underline{x}}(k|k)\,|\,Y_0(m)\} = E\{\underline{x}'(k)\underline{M}\,\underline{x}(k)\,|\,Y_0(m)\} - \mathrm{tr}\,\underline{M}\,\underline{\Sigma}(k|k) \tag{5.7.4}$$

Proof:

$$E\{\bar{\underline{x}}'(k|k)\underline{M}\,\bar{\underline{x}}(k|k)\,|\,Y_0(m)\} = E\{E\{\bar{\underline{x}}'(k|k)\underline{M}\,\bar{\underline{x}}(k|k)\,|\,Y_0(k)\}\,|\,Y_0(m)\} \tag{5.7.5}$$

But

$$E\{\bar{\underline{x}}'(k|k)\underline{M}\,\bar{\underline{x}}(k|k)\,|\,Y_0(k)\} = E\{\underline{x}'(k)\underline{M}\,\underline{x}(k)\,|\,Y_0(k)\} - \mathrm{tr}\,\underline{M}\,\underline{\Sigma}(k|k) \tag{5.7.6}$$

Since $\underline{\Sigma}(k|k)$ is independent of measurements

$$E\{\bar{\underline{x}}'(k|k)\underline{M}\,\bar{\underline{x}}(k|k)\,|\,Y_0(m)\} = E\{\underline{x}'(k)\underline{M}\,\underline{x}(k)\,|\,Y_0(m)\} - \mathrm{tr}\,\underline{M}\,\underline{\Sigma}(k|k) \tag{5.7.7}$$

Q.E.D.

The following theorem is related to the maximization problem of the coordinator.

<u>Theorem 5.7.2</u>: Let $J^*(\underline{p})$ denote the optimal cost for Problem 5.4. Then

(1) $$\max_{\underline{p}} J^*(\underline{p}) = \overline{\underline{x}}'(0)\underline{K}(0)\overline{\underline{x}}(0) + c(0) \tag{5.7.8}$$

where

$$c(0) = \sum_{i=1}^{N} c_i(0)$$

$$c_i(0) = \text{tr}\ \underline{K}_i(0)\underline{\Sigma}_i(0|0) + \sum_{k=0}^{T-1} \text{tr}(\underline{Q}_i + \underline{A}_{ii}'\underline{K}_i(k+1)\underline{A}_{ii} - \underline{K}_i(k))\underline{\Sigma}_i(k|k) + \text{tr}\ \underline{K}_i(k+1)\underline{\Xi}_i(k) \tag{5.7.9}$$

(2) the optimal $\underline{p}^*(k)$ is equal to $-\underline{\lambda}^*(k+1)$ which corresponds to the costate of the optimal deterministic control problem for the entire system

(3) $$\underline{u}_i^*(k) = -\underline{D}_i(k+1)(\hat{\underline{x}}_i(k) - \overline{\underline{x}}_i(k)) - [\underline{T}^{-1}(k+1)\underline{B}'\underline{K}(k+1)\underline{A}\ \overline{\underline{x}}(k)]_i \tag{5.7.10}$$

<u>Proof</u>: From Theorems 5.6.2 and 5.6.3, the optimal cost is given by

$$J^*(\underline{p}) = \sum_{i=1}^{N} \overline{\underline{x}}_i'(0)\underline{K}_i(0)\overline{\underline{x}}_i(0) + 2\underline{r}_i'(0)\overline{\underline{x}}_i(0) + s_i(0) - \underline{r}_i'(T)\overline{\underline{x}}_i(T)$$

$$= \overline{\underline{x}}'(0)\tilde{\underline{K}}(0)\overline{\underline{x}}(0) + 2\underline{r}'(0)\overline{\underline{x}}(0) + \sum_{i=1}^{N} s_i(0) + \underline{r}'(T)\tilde{\underline{K}}^{-1}(T)[\underline{r}(T) + \tfrac{1}{2}\underline{p}(T-1)] \tag{5.7.11}$$

$s_i(0)$ is given by equation (4.5.6), and $\underline{r}(T)$ is given by (5.6.16).

If we retain the terms involving $\underline{p}$, we have the following constrained maximization problem, ($\lambda(k)$ is as defined in Chapter 4).

$$\text{Max} \quad \underline{\lambda}'(1)\underline{A}\,\underline{\bar{x}}(0) + \sum_{k=1}^{T-1} - \underline{r}'(k)\underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k)\underline{r}(k)$$
$$+ \underline{\lambda}'(k)\underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k)\underline{r}(k)$$
$$- \frac{1}{4}\,\underline{\lambda}'(k)[\underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k) + \underline{B}\,\underline{R}^{-1}\underline{B}']\underline{\lambda}(k)$$
$$- \frac{1}{4}\,\underline{\lambda}'(T)[\underline{\tilde{S}}^{-1}(T) - \underline{\tilde{K}}^{-1}(T) + \underline{K}^{-1}(T)]\underline{\lambda}(T) \tag{5.7.12}$$

with respect to $\underline{r}(k)$; $k=1,\ldots,T-1$

$\underline{\lambda}(k)$; $k=1,\ldots,T$

Such that

$$\underline{Q}\,\underline{\tilde{K}}^{-1}(k)\underline{r}(k) = \frac{1}{2}\,\underline{A}'\underline{\lambda}(k+1) - \frac{1}{2}\,(\underline{I} - \underline{Q}\,\underline{\tilde{K}}^{-1}(k))\underline{\lambda}(k) \qquad k=1,\ldots,T-1 \tag{5.7.13}$$

Carrying out the maximization using Lagrange multipliers, we obtain the same equations as in Theorem 4.5.1 except equation (4.5.15) which now becomes

$$\frac{1}{2}\,\underline{A}\,\underline{\alpha}^*(T-1) - \frac{1}{2}\,(\underline{\tilde{S}}^{-1}(T) - \underline{\tilde{K}}^{-1}(T) + \underline{K}^{-1}(T))\underline{\lambda}^*(T) = 0 \tag{5.5.14}$$

Since

$$\underline{\tilde{S}}^{-1}(T) = \underline{\tilde{K}}^{-1}(T) + \underline{B}\,\underline{R}^{-1}\underline{B}' \tag{5.7.15}$$

we have

$$\underline{A}\,\underline{\alpha}^*(T-1) - (\underline{B}\,\underline{R}^{-1}\underline{B}' + \underline{K}^{-1}(T))\underline{\lambda}^*(T) = \underline{0}$$

or

$$\underline{\lambda}^*(T) = \underline{K}(T)[\underline{A}\,\underline{\alpha}^*(T-1) - \underline{B}\,\underline{R}^{-1}\underline{B}'\underline{\lambda}^*(T)] \tag{5.7.16}$$

This, together with $\underline{\lambda}^*(k)$, $k=1,\ldots,T-1$, are the costates corresponding to the optimal deterministic control problem for the entire system with initial state $\overline{\underline{x}}(0)$.

It can be verified (as in Section 4.5) that $\overline{\underline{x}}_i(k)$ given in equation (5.6.10) is the unconditional (a priori) estimate by the coordinator. Equation (5.6.10) then becomes

$$\underline{u}_i^*(k) = -\underline{D}_i(k+1)(\hat{\underline{x}}_i(k) - \overline{\underline{x}}_i(k)) - [\underline{T}^{-1}(k+1)\underline{B}'\underline{K}(k+1)\underline{A}\,\overline{\underline{x}}(k)]_i \tag{5.7.17}$$

$J^*(\underline{p})$ is given by equation (5.7.11). Substituting in the optimal values for $\underline{p}(0)$, we obtain

$$\begin{aligned}\underline{r}(0) &= -\frac{1}{2}\underline{A}'\underline{p}(0) - (\underline{I} - \underline{Q}\,\tilde{\underline{K}}^{-1}(0))\tilde{\underline{K}}(0)\overline{\underline{x}}(0)\\ &= (\underline{K}(0) - \tilde{\underline{K}}(0))\overline{\underline{x}}(0)\end{aligned} \tag{5.7.18}$$

From Appendix C

$$\begin{aligned}\sum_{i=1}^{N} s_i(0) + \underline{r}'(T)\tilde{\underline{K}}^{-1}(T)[\underline{r}(T) - \frac{1}{2}\underline{\lambda}^*(T)] &= c(0) + \overline{\underline{x}}'(0)\tilde{\underline{K}}(0)\overline{\underline{x}}(0)\\ &\quad - \overline{\underline{x}}'(0)\underline{K}(0)\overline{\underline{x}}(0)\end{aligned} \tag{5.7.19}$$

Therefore

$$\begin{aligned}J^*(\underline{p}^*) &= \max_{\underline{p}} J^*(p)\\ &= \overline{\underline{x}}'(0)[\tilde{\underline{K}}(0) + 2\underline{K}(0) - 2\tilde{\underline{K}}(0) - \tilde{\underline{K}}(0) - \underline{K}(0)]\overline{\underline{x}}(0) + c(0)\\ &= \overline{\underline{x}}'(0)\underline{K}(0)\overline{\underline{x}}(0) + c(0)\end{aligned} \tag{5.7.20}$$

Q.E.D.

With Lemma 5.7.1 and Theorem 5.7.2, equation 5.7.1 can now be solved.

Theorem 5.7.3: The solution to equation (5.7.1) is given by

$$V(Y_0(k),k) = \underline{\bar{x}}'(k|k)\underline{K}(k)\underline{\bar{x}}(k|k) + b(k)$$

$$k = 0,\ell,\ldots,T-\ell \qquad (5.7.21)$$

where $\underline{\bar{x}}(k|k)$ is the estimate of the state of the system by the coordinator given by equation (5.4.7).

$\underline{K}(k)$ is the solution to the Riccati equation for the entire system.

$$b(k) = b(k+\ell) - \mathrm{tr}\ \underline{K}(k+\ell)\underline{\Sigma}(k+\ell|k+\ell) + c(k) \qquad (5.7.22)$$

$$c(k) = \sum_{i=1}^{N}\Big\{\mathrm{tr}\ \underline{K}_i(k|k+\ell)\underline{\Sigma}_i(k|k) + \sum_{t=k}^{k+\ell-1}\mathrm{tr}(\underline{Q}_i + \underline{A}_{ii}'\underline{K}_i(t+1;k+\ell)\underline{A}_{ii}$$

$$- \underline{K}_i(t;k+\ell))\underline{\Sigma}_i(t|t) + \mathrm{tr}\ \underline{K}_i(t+1;k+\ell)\underline{\Xi}_i(t)\Big\}$$

$$(5.7.23)$$

$\underline{K}_i(t;k+\ell)$ is given by

$$\underline{K}_i(t;k+\ell) = \underline{Q}_i + \underline{A}_{ii}'\underline{K}_i(t+1;k+\ell)[\underline{I} - \underline{B}_i\underline{T}_i^{-1}(t+1;k+\ell)\underline{B}_i'\underline{K}_i(t+1;k+\ell)]\underline{A}_{ii}$$

$$\underline{K}_i(k+\ell;k+\ell) = \underline{K}_{ii}(k+\ell) \qquad (5.7.24)$$

$$\underline{T}_i(t+1;k+\ell) = \underline{R}_i + \underline{B}_i'\underline{K}_i(t+1;k+\ell)\underline{B}_i \qquad (5.7.25)$$

$$b(T) = \mathrm{tr}\ \underline{F}\ \underline{\Sigma}\ (T|T) \qquad (5.7.26)$$

Proof: To solve for $V(Y_0(T-\ell),T-\ell)$ we use Theorem 5.7.2, using the statistics generated from $Y_0(T-\ell)$ as the a priori information. From equation (5.7.8) we thus obtain

$$V(Y_0(T-\ell),T-\ell) = \underline{\bar{x}}'(T-\ell|T-\ell)\underline{K}(T-\ell)\underline{\bar{x}}(T-\ell|T-\ell) + b(T-\ell) \qquad (5.7.27)$$

Suppose

$$V(Y_0(k+\ell),\ k+\ell) = \underline{\bar{x}}'(k+\ell|k+\ell)\underline{K}(k+\ell)\underline{\bar{x}}(k+\ell|k+\ell) + b(k) \tag{5.7.28}$$

Then, using Lemma 5.7.1, equation (5.7.1) becomes

$$V(Y_0(k),k) = \underset{\underline{p}(k),\dots,\underline{p}(k+\ell-1)}{\mathrm{Max}} \quad \underset{\substack{\underline{\gamma}^k,\dots,\underline{\gamma}^{k+\ell-1} \\ \underline{v}(k),\dots,\underline{v}(k+\ell-1)}}{\mathrm{Min}}$$

$$E\Big\{\sum_{i=1}^{N}\sum_{t=k}^{k+\ell-1} \underline{x}_i'(t)\underline{Q}_i\underline{x}_i(t) + \underline{u}_i'(t)\underline{R}_i\underline{u}_i(t) + \underline{p}_i'(t)\underline{v}_i(t) - \underline{\tilde{p}}_i'(t)\underline{x}_i(t)$$

$$+ \underline{x}'(k+\ell)\underline{K}(k+\ell)\underline{x}(k+\ell)\,\big|\,Y_0(k)\Big\}$$

$$+ b(k) - \mathrm{tr}\ \underline{K}(k+\ell)\underline{\Sigma}(k+\ell|k+\ell) \tag{5.7.29}$$

From Theorem 5.7.2, with the common a priori statistics generated by $Y_0(k)$, we thus have equations (5.7.21) to (5.7.25). (Q.E.D.)

We have demonstrated that at each updating time k, the lower level controllers need only to solve a stochastic control problem up to the next updating time $k+\ell$. However, this problem is not uncoupled into N sub-problems even with the coordinating parameters $\underline{p}$ supplied by the coordinator. To uncouple the sub-problems, the coordinator has to send out an extra signal $\underline{r}_i(k+\ell|k)$ given by

$$\underline{r}_i(k+\ell|k) = \sum_{j\neq i} \underline{K}_{ij}(k+\ell)\underline{\bar{x}}_j(k+\ell|k) \tag{5.7.30}$$

where

$$\underline{\bar{x}}(k+\ell|k) = E\{\ \underline{x}(k+\ell)\,|\ Y_0(k)\} \tag{5.7.31}$$

and can be generated from equation (5.4.10).

Specifically, at the updating time k, the ith controller faces an uncoupled problem with the following cost functional.

$$\tilde{J}_i(Y_0(k),k) = E\Big\{\sum_{t=k}^{k+\ell-1} \underline{x}_i'(t)\underline{Q}_i\underline{x}_i(t) + \underline{u}_i'(t)\underline{R}_i\underline{u}_i(t) + \underline{p}_i^{*\prime}(t;k)\underline{v}_i(t)$$

$$- \tilde{\underline{p}}_i^{*\prime}(t;k)\underline{x}_i(t) + \underline{x}_i'(k+\ell)\underline{K}_{ii}(k+\ell)\underline{x}_i(k+\ell)$$

$$+ 2\,\underline{r}_i'(k+\ell|k)\underline{x}_i(k+\ell)\,\Big|\,Y_0(k)\Big\} \qquad (5.7.31)$$

The coordinator generates $\overline{\underline{x}}(k|k)$, $\underline{\Sigma}(k|k)$, $\underline{r}(k+\ell|k)$ and $\underline{p}^*(t;k)$, $t = k,\ldots,k+\ell-1$ using equations (5.4.6), (5.7.30) and (5.4.9) and his data $Y_0(k)$. Moreover, $\underline{K}_{ii}(k+\ell)$ is also required by the ith controller. These are transmitted to the lower level to define their uncoupled decision problems specified by equation (5.7.31).

Using the results of Section 6, the optimal controls for the lower level are found to be the following.

For $k \leq t < k+\ell$

$$\underline{u}_i^*(t) = -\underline{D}_i(t+1;k+\ell)(\hat{\underline{x}}_i(t|t) - \overline{\underline{x}}_i(t|k)) - \underline{E}_i(t+1)\underline{p}_i^*(t;k)$$

$$(5.7.32)$$

where $\underline{E}_i(t+1)$ is the same as that given in Chapter 4.

$$\underline{D}_i(t+1;k+\ell) = \underline{T}_i^{-1}(t+1;k+\ell)\underline{B}_i'\underline{K}_i(t+1;k+\ell)\underline{A}_{ii} \qquad (5.7.33)$$

where $\underline{T}_i(t+1;k+\ell)$ and $\underline{K}_i(t+1;k+\ell)$ are given by equations (5.7.24) and (5.7.25).

$\hat{\underline{x}}_i(t|t)$ and $\overline{\underline{x}}_i(t|k)$ have the same interpretation as in Section 4. $\underline{p}_i^*(t;k)$ is given by equation (5.4.9). Comparing with equation (5.4.3) we see that equatinn (5.7.2) differs only in the gain matrix $\underline{D}_i(t+1;k+\ell)$.

The part depending only on the coordinating parameters remains the same. When $\underline{\hat{x}}_i(t|t) - \underline{\bar{x}}_i(t|k)$ is zero, then the coordinator takes over the control completely and it is the same as that given by a separation theorem (recall the result of Section 4). Thus asymptotically both open loop feedback optimal coordination and closed loop optimal coordination give the same results.

Summarizing, closed loop optimal coordination differs from open loop feedback optimal coordination in two respects.

(1) The lower level problems are easier to solve since the time interval under consideration is shorter.

(2) The coordinator has to take into consideration the fact that he will be gathering information in the future. This results in a more sophisticated task of coordination. Apart from the usual coordinating parameter $\underline{p}$ which has to be transmitted and the state estimates, the coordinator has to give each lower level controller both $\underline{K}_{ii}(k+\ell)$ and $\underline{r}_i(k+\ell|k)$.

8. Discussion and Perspectives

In this chapter we have studied the on-line coordination of dynamic systems when the coordinator collects measurements from the lower level periodically. Two types of on-line coordination have been considered: open loop feedback optimal and closed loop optimal. Open loop feedback optimal coordination is conceptually simple and ignores the availability of future measurements to the coordinator. Essentially the coordinator and the lower level controllers have to solve an off-line coordination problem at each updating time. For the linear-quadratic-Gaussian case simple control strategies are obtained which have nice physical interpretations. The control strategy of each local controller has two parts: a part which depends on the difference between his local estimate and the coordinator's estimate about the state of his subsystem, and a part which depends on the information of the coordinator.

Closed loop coordination assumes the availability of future measurements to the coordinator. In general a functional equation has to be solved by the coordinator. Even with the optimal coordinating parameters, the lower level problems are not uncoupled between updating times. In the linear-quadratic-Gaussian case, these lower level problems can be decomposed by the introduction of additional coordinating parameters. The resulting optimal control strategies are very similar to those obtained in open loop feedback optimal coordination. In fact, only the gain matrices multiplying the difference of the state estimates are different.

CHAPTER 6

CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

In this thesis we have investigated ways of controlling a large-scale system in a decentralized manner. The two important aspects of computation and information are considered simultaneously. It is found that for systems with a particular structure, control strategies which utilize decentralized information can also be obtained by decentralized computation.

In some problems, such as the static optimization problem considered in Chapter 2, the systems have this nice structure and control strategies which are computationally and informationally decentralized can be obtained right away. In some other problems, like those coupled dynamic systems that we considered in Chapter 3, 4, and 5, this nice structure is not inherent. However, by considering another kind of optimality, the problem can be reformulated to have that structure. It is then possible to identify two levels of information structure, one belonging to the lower level controllers and one belonging to the coordinator who sees that certain constraints are satisfied. The problem can be solved in a two-level scheme. The lower level problems are decomposed both informationally and computationally when optimal coordinating parameters are transmitted from the coordinator. The job of the coordinator is to find these optimal coordinating parameters.

The appeal of this approach lies in the ease with which the optimal strategies can be found. A high dimensional stochastic

control problem is reduced to a number of lower dimensional problems. Even for nonlinear problems, if the solutions to the lower level problems are known, the solution for the entire problem can be constructed. By specializing to the linear-quadratic-Gaussian case, we obtain control strategies which are physically intuitive.

We have also allowed the situation when the coordinator makes measurements on-line. The control strategies obtained in Chapter 5 for open loop feedback optimal coordination and closed loop optimal coordination provide alternative solutions to dynamic team problems when some sharing of past information is allowed.

The whole area of research in the decentralized control of large-scale systems is still wide open. Some areas which arise immediately from this thesis are the following. In the static optimization of stochastic systems, more work should be done to relate the information available to the different decision agents and their subsystems such that decomposition is possible. Results for the existence of a decomposition and computational algorithms to arrive at the decomposition are also desirable.

In dynamic systems, results are needed in comparing the kind of optimality defined in this thesis with the usual kind of stochastic optimality. Optimality seems to be related to how well the constraint is satisfied. Intuitively, we would expect a system to be more optimal when the constraints are satisfied more exactly. It should be possible to define a degree of coordination based on this constraint such that optimality is related to the degree of coordination.

In this thesis, the coordinator has complete a priori information about the structure of the entire system. This leads to a rather complicated decision task for the coordinator. An area for future research will be the aggregation of a priori information and its relation to performance.

APPENDIX A

SOME RESULTS IN PROBABILITY THEORY

In this appendix we summarize some definitions and results in probability theory which have been used in this thesis. The probability space under consideration is denoted by $(X, \mathcal{B}, \mu)$. $\mathcal{F}$ and $\mathcal{F}_0$ are sub-σ-fields of $\mathcal{B}$.

Def. A.1: $\mathcal{F} \cap \mathcal{F}_0$ is the smallest-σ-field generated by $A \cap B$, where $A \in \mathcal{F}$ and $B \in \mathcal{F}_0$.

Lemma A.1: For any random variable ℓ, if $E\{\ell|\mathcal{F}_0\}$ is measurable with respect to $\mathcal{G}$, $\mathcal{G} \subset \mathcal{F}_0$, then $E\{\ell|\mathcal{F}_0\} = E\{\ell|\mathcal{G}\}$ a.e.

Proof: Given any random variable ℓ and a σ-subfield $\mathcal{G}$, the conditional expectation $E\{\ell|\mathcal{G}\}$ is characterized by two conditions:

(a) It is measurable with respect to $\mathcal{G}$;

(b) $$\int_A E\{\ell|\mathcal{G}\} d\mu = \int_A \ell \, d\mu \quad \text{for every } A \in \mathcal{G} \tag{A.1}$$

$E\{\ell|\mathcal{F}_0\}$ is measurable with respect to $\mathcal{G}$. Moreover,

$$\int_B E\{\ell|\mathcal{F}_0\} d\mu = \int_B \ell \, d\mu \quad \text{for every } B \in \mathcal{F}_0 \tag{A.2}$$

Since $\mathcal{G} \subset \mathcal{F}_0$, (A.2) is also true for every $B \in \mathcal{G}$

Thus $E\{\ell|\mathcal{F}_0\}$ satisfies equation (A.1), and $E\{\ell|\mathcal{F}_0\} = E\{\ell|\mathcal{G}\}$ a.e.

Q.E.D.

<u>Lemma A.2</u>: Let γ be a $F \cap F_0$ - measurable function from X into U. Let f be a measurable real-valued function on U x X. Then given any $y \varepsilon X$, there exists a function $\gamma(.;y)$ measurable with respect to F such that

$$E\{f(\gamma(x),x)|F_0\}(y) = E\{f(\gamma(x;y),x)|F_0\}(y) \quad \text{a.e.} \tag{A.3}$$

<u>Proof</u>: We assume two conditions, which, for this thesis, will be satisfied.

(1) There exists a regular conditional probability measure $P_y^{\circ}(A)$.

(2) F and F_0 are fields generated by functions h and h_0 so that γ being $F \cap F_0$ - measurable is equivalent to

$$\gamma(x) = \eta(h(x),\ h_0(x)) \tag{A.4}$$

where η is $A \times A_0$ - measurable on $Z \times Z_0$

$h : X \to Z$

$h_0 : X \to Z_0$

A and A_0 are σ-fields on Z and Z_0

Let $\gamma(x;y) = \eta(h(x),h_0(y))$. Then given y, $\gamma(.;y)$ is F - measurable.

$$\begin{aligned} E\{f(\gamma(x),x)|F_0\}(y) &= \int_X f(\eta(h(x),h_0(x)),x)d\, P_y^{\circ}(x) \\ &= \int_A f(\eta(h(x),h_0(x)),x)d\, P_y^{\circ}(x) \\ &\quad + \int_{X-A} f(y(h(x),h_0(x)),x)d\, P_y^{\circ}(x) \end{aligned} \tag{A.5}$$

where

$$A = \{x;h_0(x) = h_0(y)\}\ \varepsilon\ F_0 \tag{A.6}$$

Given $A \in F_0$, for all $B \in F_0$ (see Ref. [L4])

$$\int_B P_y^{\circ}(A)\,d\,\mu(y) = \mu(A \cap B) = \int_B 1_A(x)\,d\,\mu(x) \qquad (A.7)$$

Therefore for all $A \in F_0$,

$$P_y^{\circ}(A) = 1_A(y) \quad \text{for almost all } y \qquad (A.8)$$

where 1_A is the indicator funtion of A.

From equation (A.6), $y \in A$. Thus

$$P_y^{\circ}(A) = 1 \quad \text{for almost all } y \qquad (A.9)$$

Equation (A.5) then becomes

$$\begin{aligned} E\{f(\gamma(x),x)|F_0\}(y) &= \int_A f(\eta(h(x),h_0(x)),x)\,d\,P_y^{\circ}(x) \\ &= \int_X f(\eta(h(x),h_0(y)),x)\,d\,P_y^{\circ}(x) \\ &= E\{f(\gamma(x;y),x)|F_0\}(y) \end{aligned} \qquad (A.10)$$

Q.E.D.

Remark: If $F = \{X, \Phi\}$ then this result reduces to the usual identity

$$E\{f(\gamma(x),x)|F_0\}(y) = E\{f(\gamma(y),x)|F_0\}(y) \qquad (A.11)$$

For a discussion of substitution in conditional expectation, see [B1].

Lemma A.3: Let $f(u,v,y,z,x)$ be a function such that x,y,z are random variables. Suppose it is desired to choose $u(y,z)$ and $v(y)$ such that $E\{f(u(y,z),v(y),y,z,x)\}$ is minimized.

Let $u^{\circ}(y,z)$, $v^{\circ}(y)$ be the minimum of

$$\min_{\substack{u \\ v(.)}} E\{f(u,v(y),y,z,x)|y,z\}$$

Then

$$\underset{\substack{u(.,.)\\ v(.)}}{\text{Min}}\ E\{f(u(y,z),v(y),y,z,x)\} = E\{f(u^\circ(y,z),v^\circ(y),y,z,x)\}$$

$$= E\{\underset{\substack{u\\ v(.)}}{\text{Min}}\ E\{f(u,v(y),y,z,x)\,|\,y,z\}\} \qquad \text{(A.12)}$$

Proof:

$$E\{f(u^\circ(y,z),v^\circ(y),y,z,x)\,|\,y,z\} \leq E\{f(u(y,z),v(y),y,z,x)\,|\,y,z\}$$

$$\text{for all } u(.,.),v(.)$$

$$\text{(A.13)}$$

Thus

$$E\{f(u^\circ(y,z),v^\circ(y),y,z,x)\} = E\{E\{f(u^\circ(y,z),v^\circ(y),y,z,x)\,|\,y,z\}\}$$

$$\leq E\{f(u(y,z),v(y),y,z,x)\} \text{ for all } u(.,.),v(.)$$

$$\text{(A.14)}$$

or

$$E\{f(u^\circ(y,z),v^\circ(y),y,z,x)\} \leq \underset{\substack{u(.,.)\\ v(.)}}{\text{Min}}\ E\{f(u(y,z),v(y),y,z,x)\}$$

$$\text{(A.15)}$$

But

$$\underset{u(.,.)}{\text{Min}}\ E\{f(u(y,z),v(y),y,z,x)\} \leq E\{f(u^\circ(y,z),v^\circ(y),y,z,x)\}$$

$$\text{(A.16)}$$

Hence we obtain equation (A.12)

Q.E.D.

APPENDIX B

INVERTIBILITY OF $\underline{K}_i(k)$ AND VERIFICATION OF EQUATION (4.4.32)

1. Invertibility of $K_i(k)$

$$\underline{K}_i(k) = \underline{Q}_i + \underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii} \qquad \underline{K}_i(T) = \underline{F}_i > \underline{0} \tag{B.1}$$

where

$$\underline{S}_i(k+1) = \underline{K}_i(k+1) - \underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1) \tag{B.2}$$

If $\underline{K}_i(k+1) > \underline{0}$, then $\underline{K}_i^{-1}(k+1)$ exists

$$\underline{S}_i(k+1) = [\underline{K}_i^{-1}(k+1) + \underline{B}_i\underline{R}_i^{-1}\underline{B}_i']^{-1} \tag{B.3}$$

$$\underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii} \geq \underline{0} \tag{B.4}$$

If $\underline{Q}_i > \underline{0}$, then $\underline{K}_i(k) > \underline{0}$

Therefore by induction $\underline{K}_i(k)$ is invertible.

Remark: $\underline{Q}_i > \underline{0}$ is sufficient but not necessary. If $\underline{A}_{ii} > \underline{0}$ then $\underline{K}_i(k) > \underline{0}$

2. Verification of Equation (4.4.32)

$$\underline{T}_i^{-1}(k+1)\underline{B}_i'(\underline{I}-\underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i')^{-1}$$

$$= \underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1)\underline{S}_i^{-1}(k+1)$$

$$= \underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1)[\underline{K}_i^{-1}(k+1) + \underline{B}_i\underline{R}_i^{-1}\underline{B}_i']$$

$$= \underline{T}_i^{-1}(k+1)[\underline{R}_i + \underline{B}_i'\underline{K}_i(k+1)\underline{B}_i]\underline{R}_i^{-1}\underline{B}_i'$$

$$= \underline{R}_i^{-1}\underline{B}_i' \tag{B.5}$$

APPENDIX C

VERIFICATION OF EQUATION (5.7.19)

1. $$\sum_{k=0}^{T-1} \mathrm{tr}\, \underline{Q}_i \underline{\Sigma}_i(k|k) + \mathrm{tr}\, \underline{K}_i(k+1)[\underline{\Sigma}_i(k+1|k) - \underline{\Sigma}_i(k+1|k+1)] + \mathrm{tr}\, \underline{K}_i(T)\underline{\Sigma}_i(T|T)$$

$$= \sum_{k=0}^{T-1} \mathrm{tr}\, \underline{Q}_i \underline{\Sigma}_i(k|k) + \mathrm{tr}\, \underline{K}_i(k+1)[\underline{A}_{ii}\underline{\Sigma}_i(k|k)\underline{A}_{ii}' - \underline{\Sigma}_i(k+1|k+1) + \underline{\Xi}_i(k)] + \mathrm{tr}\, \underline{K}_i(T)\underline{\Sigma}_i(T|T)$$

$$= \sum_{k=0}^{T-1} \mathrm{tr}(\underline{Q}_i + \underline{A}_{ii}'\underline{K}_i(k+1)\underline{A}_{ii})\underline{\Sigma}_i(k|k) + \mathrm{tr}\, \underline{K}_i(k+1)\underline{\Xi}_i(k) - \mathrm{tr}\, \underline{K}_i(k+1)\underline{\Sigma}_i(k+1|k+1) + \mathrm{tr}\, \underline{K}_i(T)\underline{\Sigma}_i(T|T) + \mathrm{tr}\, \underline{K}_i(0)\underline{\Sigma}_i(0|0) - \mathrm{tr}\, \underline{K}_i(0)\underline{\Sigma}_i(0|0)$$

$$= \mathrm{tr}\, \underline{K}_i(0)\underline{\Sigma}_i(0|0) + \sum_{k=0}^{T-1} \mathrm{tr}(\underline{Q}_i + \underline{A}_{ii}'\underline{K}_i(k+1)\underline{A}_{ii} - \underline{K}_i(k))\underline{\Sigma}_i(k|k) + \mathrm{tr}\, \underline{K}_i(k+1)\underline{\Xi}_i(k) \qquad \text{(C.1)}$$

$$= c_i(0)$$

2. From equation 4.5.6,

$$\sum_{i=1}^{N} s_i(0) = \underline{\bar{x}}'(0)\underline{\tilde{A}}'\underline{\tilde{S}}(1)\underline{\tilde{A}}\,\underline{\bar{x}}(0) + c(0) + \sum_{k=1}^{T-1} - \underline{r}'(k)\underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k)\underline{r}(k) + \underline{\lambda}^{*\prime}(k)\underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k)\underline{r}(k) - \frac{1}{4}\underline{\lambda}^{*\prime}(k)[\underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k) + \underline{B}\,\underline{R}^{-1}B']\underline{\lambda}^*(k) - \frac{1}{4}\underline{\lambda}^{*\prime}(T)\underline{\tilde{S}}^{-1}(T)\underline{\lambda}^*(T) - \underline{r}'(T)\underline{\tilde{K}}^{-1}(T)\underline{r}(T) + \underline{\lambda}^{*\prime}(T)\underline{\tilde{K}}^{-1}(T)\underline{r}(T) \qquad \text{(C.2)}$$

From equation (4.4.37),

$$\underline{r}(k) = -\tilde{\underline{K}}(k)\overline{\underline{x}}(k) + \frac{1}{2}\underline{\lambda}^*(k) \qquad k=1,\ldots,T-1 \tag{C.3}$$

From equation (5.6.16)

$$\underline{r}(T) = -\frac{1}{2}\underline{\lambda}^*(T) - \frac{1}{2}\tilde{\underline{K}}(T)\underline{K}^{-1}(T)\underline{\lambda}^*(T) \tag{C.4}$$

Thus,

$$\sum_{i=1}^{N} s_i(0) + \underline{r}'(T)\tilde{\underline{K}}^{-1}(T)[\underline{r}(T) - \frac{1}{2}\underline{\lambda}^*(T)]$$

$$= c(0) + \overline{\underline{x}}'(0)\tilde{\underline{A}}'\tilde{\underline{S}}(1)\tilde{\underline{A}}\,\overline{\underline{x}}(0)$$

$$- \sum_{k=1}^{T-1}\{\overline{\underline{x}}'(k)\underline{Q}\,\overline{\underline{x}}(k) + \frac{1}{4}\underline{\lambda}^{*\prime}(k)\underline{B}\,\underline{R}^{-1}\underline{B}'\underline{\lambda}^*(k)\}$$

$$- \frac{1}{4}\underline{\lambda}^{*\prime}(T)[\underline{B}\,\underline{R}^{-1}\underline{B}' + \underline{K}^{-1}(T)]\underline{\lambda}^*(T) \tag{C.5}$$

But

$$\tilde{\underline{A}}'\tilde{\underline{S}}(1)\tilde{\underline{A}} = \tilde{\underline{K}}(0) - \underline{Q} \tag{C.6}$$

$$\frac{1}{4}\underline{\lambda}^{*\prime}(k)\underline{B}\,\underline{R}^{-1}\underline{B}'\underline{\lambda}^*(k) = (\frac{1}{2}\underline{R}^{-1}\underline{B}'\underline{\lambda}^*(k))'\underline{R}(\frac{1}{2}\underline{R}^{-1}\underline{B}'\underline{\lambda}^*(k))$$

$$= \underline{u}^{*\prime}(k-1)\underline{R}\,\underline{u}^*(k-1) \quad k=1,\ldots T \tag{C.7}$$

$$\frac{1}{4}\underline{\lambda}^{*\prime}(T)\underline{K}^{-1}(T)\underline{\lambda}^*(T) = (\frac{1}{2}\underline{K}^{-1}(T)\underline{\lambda}^*(T))'\underline{K}(T)(\frac{1}{2}\underline{K}^{-1}(T)\underline{\lambda}^*(T))$$

$$= \overline{\underline{x}}'(T)\underline{K}(T)\overline{\underline{x}}(T) \tag{C.8}$$

where $\underline{u}^*(k)$ is the optimal control with initial state $\overline{\underline{x}}(0)$.

Therefore

$$\sum_{i=1}^{N} s_i(0) + \underline{r}'(T)\tilde{\underline{K}}^{-1}(T)[\underline{r}(T) - \frac{1}{2}\underline{\lambda}^*(T)]$$

$$= c(0) + \overline{\underline{x}}'(0)\tilde{\underline{K}}(0)\overline{\underline{x}}(0) - \sum_{k=0}^{T-1} \overline{\underline{x}}'(k)\underline{Q}\,\overline{\underline{x}}(k)$$

$$+ \underline{u}^{*\prime}(k)\underline{R}\,\underline{u}^*(k) + \overline{\underline{x}}'(T)\underline{K}(T)\overline{\underline{x}}(T)$$

$$= c(0) + \overline{\underline{x}}'(0)\tilde{\underline{K}}(0)\overline{\underline{x}}(0) - \overline{\underline{x}}'(0)\underline{K}(0)\overline{\underline{x}}(0)$$

$$= c(0) + \overline{\underline{x}}'(0)[\tilde{\underline{K}}(0) - \underline{K}(0)]\overline{\underline{x}}(0) \qquad \text{(C.9)}$$

Q.E.D.

APPENDIX D

VERIFICATION OF EQUATION (5.6.22)

$$\underline{u}^*(k) = -\frac{1}{2}\underline{R}^{-1}\underline{B}'\underline{p}^*(k;k) \tag{D.1}$$

From (5.4.9)

$$\begin{aligned}\underline{p}^*(k;k) &= -2\,\underline{K}(k+1)\bar{\underline{x}}(k+1|k)\\ &= -2\,\underline{K}(k+1)[\underline{A}\,\bar{\underline{x}}(k|k) + \underline{B}\,\underline{u}^*(k)]\end{aligned} \tag{D.2}$$

Therefore

$$\underline{R}\,\underline{u}^*(k) = \underline{B}'\underline{K}(k+1)[\underline{A}\,\bar{\underline{x}}(k|k) + \underline{B}\,\underline{u}^*(k)] \tag{D.3}$$

or

$$u^*(k) = \underline{T}^{-1}(k+1)\underline{B}'\underline{K}(k+1)\underline{A}\,\bar{\underline{x}}(k|k) \tag{D.4}$$

Q.E.D.

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